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## Ladder system uniformization on trees

Uniformization properties of ladder system colourings have been extensively studied due to various connections to algebra, in particular to the Whitehead problem and its relatives [4, 7, 8], to topology [1, 3, 9, 13], and to fundamental questions in combinatorial set theory [5, 6, 10].

Our current interest lies in understanding a relatively new version of the uniformization property introduced by Justin Moore, a notion that played a key role in understanding minimal uncountable linear orders [6]. Given a tree $T$ of height $\omega_1$, we say that a ladder system colouring $(f_\alpha)_{\alpha\in\lim\omega_1}$ has a $T$-uniformization if there is a function $\varphi$ defined on a pruned, downward closed subtree $S$ of $T$ so that for any $s \in S_\alpha$ of limit height and almost all $\xi \in \mathrm{dom} f_\alpha$, $\varphi(s \restriction \xi) = f_\alpha(\xi)$.

In this talk, I will first outline how variations of the diamond principle (weak and strong) imply that there are ladder system colourings without $T$-uniformizations; as expected, stronger diamonds imply that we can take care of more trees and find simple colourings without $T$-uniformizations. However, rather unexpectedly, we prove that whenever $\diamondsuit^+$ holds, for any ladder system $\mathbf{C}$ there is an Aronszajn tree $T$ so that any monochromatic colouring of $\mathbf{C}$ (i.e. each $f_\alpha$ is some constant function) has a $T$-uniformization [11].

Second, we look at the existence of $T$-uniformizations for Suslin trees $T$ and answering a question of J. Baumgartner [2], I outline why the existence of a Suslin tree does not necessarily imply that there are minimal uncountable linear orders other than $\omega_1$ and its reverse [12]. It remains open from [2] if $\diamondsuit$ suffices to construct such minimal linear orders.

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