Side conditions, adding few reals, and trees

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The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms together with CH has a long history, starting with Jensen's landmark result that Suslin's Hypothesis is compatible with GCH.

Most of the work in the area done so far proceeds by showing that some suitable countable support iteration whose iterands are proper forcing notions not adding new reals fails to add new reals at limit stages.

There are (nontrivial) limitations to what can be achieved in this area. In fact, there cannot be any 'master' iteration lemma:

A.–Larson–Moore: Modulo a mild large cardinal assumption, there are two Π_2 statements over $H(\omega_2)$, each of which can be forced, using proper forcing, to
hold together with CH, and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

Above result closely tied to the following concrete well–known obstacle to not adding reals: Given a ladder system $\vec{C} = (C_\delta \,:\, \delta \in Lim(\omega_1))$, let $\mathrm{Unif}(\vec{C})$ denote the statement that for every colouring $F : \mathrm{Lim}(\omega_1) \longrightarrow \{0, 1\}$ there is $G : \omega_1 \longrightarrow \{0, 1\}$ such that that for every $\delta \in \mathrm{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \setminus \alpha$. We say that G uniformizes F on $\vec{C}$.

Given $\vec{C}$ and F as above there is a natural forcing notion, $\mathcal{Q}_{\vec{C},F}$, for adding a uniformizing function for F on $\vec{C}$ by initial segments. Easy to see that $\mathcal{Q}_{\vec{C},F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $\mathcal{Q}_{\vec{C},F}$, even with a fixed $\vec{C}$, will necessarily add new reals. In fact, the existence of a ladder system $\vec{C}$ for which $\mathrm{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$ (Devlin–Shelah).

Proof: Fix a bijection $h : \omega \longrightarrow \omega \times \omega$ such that $i \leq n$ if $h(n+1) = (i, j)$. For each $g : \omega_1 \longrightarrow 2$ construct $f_n : \omega_1 \longrightarrow 2$ $(n < \omega)$ such that

$$f_0 = g$$

and

$$f_{n+1} \restriction C_\delta =_{\text{fin}} f_i(\delta + j)$$

for every limit $\delta \neq 0$, where $h(n+1) = (i, j)$.
Given f_k $(k \leq n)$, f_{n+1} exists by applying $\text{Unif}(\vec{C})$ to the colouring

$$\delta \longrightarrow f_i(\delta + j)$$

But now, for each limit $\delta \neq 0$, $(f_n \restriction \delta \;:\; n < \omega)$ determines $(f_n \restriction \delta + \omega \;:\; n < \omega)$. Hence,

$$(f_n \restriction \omega \;:\; n < \omega)$$

determines

$$(f_n \;:\; n < \omega),$$

and in particular $f_0 = g$. Hence $2^{\aleph_0} = 2^{\aleph_1}$. $\square$

Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

We say that C *measures* $\vec{C}$.

Measuring implies $\neg$WCG: Suppose $\vec{C} = (C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ ladder system and $C \subseteq \omega_1$ is a club measuring $\vec{C}$. Then, for every $\delta \in C$, if δ is a limit point of limit points of C, then a tail of $C \cap \delta$ is disjoint from C_δ since $\mathrm{ot}(C_\delta) = \omega$.

Natural forcing for adding a club measuring a given $\vec{C}$ by initial segments is proper and adds no new reals. On the other hand it is not known if these forcings can (consistently) be iterated without adding new reals. Strongest failures of Club–Guessing known to be within reach of current techniques for iterating proper forcing without adding reals are in the region of $\neg$WCG (Shelah, NNR revisited).

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(Moore) Is Measuring consistent with CH?

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In joint work with Mota, we addressed Moore's question. In order to do so we distanced ourselves from the tradition of preserving CH by not adding reals; we aimed at building interesting models of CH by a cardinal–preserving forcing which actually adds reals (but only $\aleph_1$–many of them).

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Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

Given a cardinal κ and $T \subseteq H(\kappa)$, a finite $\mathcal{N} \subseteq [H(\kappa)]^{\aleph_0}$ is a *T–symmetric system* if

(1) for every $N \in \mathcal{N}$,

$$(N, \in, T) \preccurlyeq (H(\kappa), \in, T),$$

(2) given N_0, $N_1 \in \mathcal{N}$, if $N_0 \cap \omega_1 = N_1 \cap \omega_1$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0, \in, T) \longrightarrow (N_1, \in, T)$$

and Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) Given N_0, $N_1 \in \mathcal{N}$ such that $N_0 \cap \omega_1 = N_1 \cap \omega_1$ and $M \in N_0 \cap \mathcal{N}$, $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

(4) Given M, $N_0 \in \mathcal{N}$ such that $M \cap \omega_1 < N_0 \cap \omega_1$, there is some $N_1 \in \mathcal{N}$ such that $N_1 \cap \omega_1 = N_0 \cap \omega_1$ and $M \in N_1$.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then $\Psi_{N,N'}$ is an isomorphism

$$\Psi_{N,N'} : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

Proof: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. **By symmetry,** $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$.
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In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

Something one may want to try at this point: Extend the symmetry requirements **also** to the working parts in such a way that the above CH–preservation argument goes trough. Hope to be able to force something interesting this way.

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Theorem

(A.–Mota) (CH) Let $\kappa > \omega_2$ be a regular cardinal such that $2^{<\kappa} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

(1) *$\mathcal{P}$ is proper and $\aleph_2$–Knaster.*

(2) *$\mathcal{P}$ forces the following statements.*

   (a) *Measuring*
   (b) *CH*
   (c) *$2^\mu = \kappa$ for every uncountable cardinal $\mu < \kappa$.*

Construction: An $\subseteq$–increasing sequence $(\mathcal{Q}_\alpha)_{\alpha \le \kappa}$ of posets.

- Each $\mathcal{Q}_\alpha$ consists of conditions $q = (f, \{(N_i, \gamma_i) \mid i < n\})$, where
  - f is a function with finite domain $\mathrm{dom}(f) \subseteq \alpha$ such that $f(\alpha)$ is a condition of suitable forcing for adding an instance to Measuring,
  - $\{N_i \mid i < n\}$ is a symmetric system,
  - γ_i is in the closure of $N \cap (\alpha + 1)$.
- Given $q = (f, \{(N_i, \gamma_i) \mid i < n\})$ and N_i, $N_{i'}$ such that $N_i \cap \omega_1 = N_{i'} \cap \omega_1$, **the natural restriction of q to N_i below γ_i is to be copied over to the natural restriction of q to $N_{i'}$ below $\gamma_{i'}$.**

The following question addresses whether or not adding reals is a necessary feature of forcing Measuring.

Question

(Moore) Does Measuring imply that there are non-constructible reals?

Trees on $\aleph_2$ and GCH

This is joint work with Mohammad Golshani.

Let κ be a regular uncountable cardinal.

- A κ–tree is a tree T of height κ all of whose levels are smaller than κ. A κ*–Aronszajn tree* is a κ–tree which has no κ–branches.

- A κ*–Souslin tree* is a κ–tree which has no κ–branches and no antichains of size κ.

- If $\kappa = \lambda^+$, a κ–Aronszajn tree T is said to be *special* if there exists a function $f : T \to \lambda$ such that $f(x) \neq f(y)$ whenever $x, y \in T$ are such that $x <_T y$. We say that f *specializes* T.

- The special Aronszajn tree property at $\kappa = \lambda^+$, $\mathrm{SATP}(\kappa)$, is the statement "there exist κ–Aronszajn trees and all such trees are special".

Aronszajn trees were introduced by Kurepa, and Aronszajn (1934) proved the existence, in ZFC, of a special $\aleph_1$–Aronszajn tree. Later, Specker (1949) showed that $2^{<\lambda} = \lambda$ implies the existence of special λ^+–Aronszajn trees for λ regular, and Jensen (1972) produced special λ^+–Aronszajn trees for singular λ in L.

Baumgartner, Malitz and Reinhardt (1970) showed that Martin's Axiom + $2^{\aleph_0} > \aleph_1$ implies SATP($\aleph_1$), and hence Souslin's Hypothesis at $\aleph_1$ as well. Later, and as already mentioned, Jensen (1974) produced a model of GCH in which SATP($\aleph_1$) holds.

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The situation at $\aleph_2$ turned out to be more complicated. Jensen (1972) proved that the existence of an $\aleph_2$–Souslin follows from each of the hypotheses $\mathsf{CH} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\})$ and $\square_{\omega_1} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\})$. The second result was improved by Gregory (1976); he proved that GCH together the existence of a non–reflecting stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ yields the existence of an $\aleph_2$–Souslin tree.

Laver and Shelah (1981) produced, relative to the existence of a weakly compact cardinal, a model of $\mathsf{ZFC} + \mathsf{CH}$ in which $\mathsf{SATP}(\aleph_2)$ holds. But in their model $2^{\aleph_1} > \aleph_2$, and the following remained a major open problem (s. e.g. Kanamori–Magidor 1977):

Question

Is ZFC+GCH consistent with the non–existence of $\aleph_2$–Souslin trees?

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In December 2017, while visiting Golshani in Tehran, we started thinking about combining the ideas from Measuring + CH with the Laver–Shelah construction for SATP($\aleph_2$). We eventually succeeded:

Theorem* (A.–Golshani) Suppose GCH holds and κ is a weakly compact cardinal. Then there exists a set–generic extension of the universe in which

(1) GCH holds,
(2) $\kappa = \aleph_2$, and
(3) SATP($\aleph_2$) holds (and hence there are no $\aleph_2$–Souslin trees).

Our argument can be easily extended to the successor of any regular cardinal.

Our large cardinal assumption is optimal:

- ⋆ Rinot (2017) proved that GCH+ Souslin's Hypothesis at $\aleph_2$ implies $\neg\square(\omega_2)$; on the other hand, Todorčević (1987) proved that $\neg\square(\omega_2)$ implies that ω_2 is weakly compact in L.

Sketch of definition of forcing

Let κ be weakly compact and assume GCH.
We define
by induction on β, a sequence $\langle \mathbb{Q}_\beta \mid \beta \leq \kappa^{++} \rangle$ of forcing notions.

Given $\beta \leq \kappa$, a condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) f_q is a countable function such that $\mathrm{dom}(f_q) \subseteq (\{0\} \cup S^{\kappa^{++}}_{\kappa}) \cap \beta$ and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.

  (a) If $\alpha = 0$, then $f_q(\alpha) \in \mathrm{Col}(\omega_1, {<}\kappa)$.
  (b) If $\alpha > 0$, then

    (i) $f_q(\alpha) : \kappa \times \omega_1 \to \omega_1$ is a countable function, and
    (ii) $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_{\beta'}$ for every $\beta' \in [\alpha, \beta)$.

(2) τ_q is a countable set of ordered pairs (N, γ), where

  (a) N is an elementary submodel of $H(\kappa^{++})$ such that ${}^{\omega}N \subseteq N$, $N \cap \kappa \in \kappa$, and $|N| = |N \cap \kappa|$,
  (b) γ is in the closure of $N \cap \beta$.
  (c) N is "ξ–sufficiently correct" for each $\xi \in N \cap \gamma$.

(3) For all $\alpha < \beta$, $q \restriction \alpha \in \mathbb{Q}_\alpha$.

(4) For all $\alpha \in \mathrm{dom}(f_q)$,
   (a) $\mathrm{cf}(\alpha) = \kappa$,
   (b) $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_{\beta'}$ for all $\beta' \in [\alpha, \beta)$, and
   (c) for all $x, y \in \mathrm{dom}(f_q(\alpha))$, if $(f_q(\alpha))(x) = (f_q(\alpha))(y)$, then $q \restriction \alpha$ does not force that x and y are comparable in $\dot{T}_\alpha$ (where $\dot{T}_\alpha$ is, in $V^{\mathbb{Q}_\alpha}$, a κ–Aronszajn tree given by a suitable book-keeping; we assume all trees are on $\kappa \times \omega_1$ with ρ-th level $\{\rho\} \times \omega_1$).

(5) Suppose $(N_0, \gamma_0), (N_1, \gamma_1) \in \tau_q$, $\alpha \in N_0 \cap \min\{\gamma_0, \gamma_1\}$, $\alpha' \in N_1 \cap \min\{\gamma_0, \gamma_1\}$, and there is an isomorphism $\Psi_{N_0,N_1} : (N_0, \in) \longrightarrow (N_1, \in)$ which
   (a) is the identity on $N_0 \cap N_1$,
   (b) is *sufficiently* correct, and such that
   (c) $\Psi_{N_0.N_1}(\alpha) = \alpha'$.

   Then the natural restriction of $q \restriction \alpha$ is isomorphic, via Ψ_{N_0,N_1}, to the natural restriction of $q \restriction \alpha'$ to N_1.

The extension relation:
Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_\beta q_0$ (q_1 is an extension of q_0) if and only if the following holds.

(A) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$

(B) $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$ for all $\alpha \in \mathrm{dom}(f_{q_0})$.

(C) $\tau_{q_0} \subseteq \tau_{q_1}$

Defining $\mathbb{Q}_{\kappa^{++}}$: Let C be the κ–club of $\beta < \kappa^{++}$ such that $\mathrm{cf}(\beta) = \kappa$ and there is some $M \preccurlyeq H(\theta)$ (θ large enough) containing $(\mathbb{Q}_\alpha)_{\alpha<\kappa^{++}}$ and such that $M \cap \kappa^{++} = \beta$.

$$\mathbb{Q}_{\kappa^{++}} = \bigcup_{\beta \in C} \mathbb{Q}_\beta$$

Main facts

(1) Due to the strong symmetry in clause (5) of the definition, it is probably not the case that $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta'}$ (or even $\mathbb{Q}_\beta \subseteq \mathbb{Q}_{\beta'}$) for all $\beta < \beta'$. On the other hand:

- $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta+1}$ for all $\beta < \kappa^{++}$.
- $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\beta'}$ for all $\beta < \beta'$ in $C \cup \{\kappa^{++}\}$.

(2) For all $\beta \leq \kappa^{++}$ such that $\mathrm{cf}(\beta) \geq \kappa$, $\mathbb{Q}_\beta$ is ω_1–strategically closed; in particular, $\mathbb{Q}_\beta$ does not add reals and hence preserves CH.

(3) $\mathbb{Q}_{\kappa^{++}}$ adds κ–many new subsets of ω_1, but not more than that; in particular, $\mathbb{Q}_{\kappa^{++}}$ preserves $2^{\aleph_1} = \aleph_2$ [essentially the same argument we saw on slide 8].

(4) If $\mathbb{Q}_{\kappa^{++}}$ has the κ–c.c. then it forces $\mathrm{SATP}(\aleph_2)$.

The κ–chain condition

Let C_0 be the set of $\beta \in \kappa^{++}$ such that

- $\mathrm{cf}(\beta) = \kappa$ and
- $\mathbb{Q}_\beta \lessdot \mathbb{Q}_{\min(C \setminus (\beta+1))}$,

and let $\tilde{C}$ be the closure of C_0 in the order topology.

Lemma

For every $\beta \in \tilde{C}$, $\mathbb{Q}_\beta$ has the κ–c.c (equivalently, it is κ–Knaster (since $\kappa \longrightarrow (\kappa)^2_2$)).

This is the most involved part of the proof, and the only place where we use the weak compactness of κ. Let $(\beta_i)_{i \leq \kappa^{++}}$ be the increasing enumeration of $\tilde{C}$ and let $\sigma = (q_\lambda)_{\lambda < \kappa}$ be a sequence of $\mathbb{Q}_{\beta_i}$-conditions. Want to find $\lambda \neq \lambda'$ so that q_λ and $q_{\lambda'}$ are compatible in $\mathbb{Q}_{\beta_i}$. The proof is by induction on i.

The case $i = 0$ is trivial ($\mathbb{Q}_{\beta_0}$ is essentially the Lévy collapse). The case when i is a limit ordinal with $\mathrm{cf}(i) < \kappa$ uses

- the fact that if two conditions q and q' are compatible in $\mathbb{Q}_\alpha$, then they have a greatest lower bound $q \oplus_\alpha q'$ (obtained essentially from closing under relevant isomorphisms Ψ_{N_0,N_1}) together with
- $\kappa \longrightarrow (\kappa)^2_{\mathrm{cf}(i)}$.

If q_λ and $q_{\lambda'}$ are incompatible then there is some $\bar{i} < i$ such that $(q_\lambda \restriction \beta_{\bar{i}}) \oplus_{\beta_{\bar{i}}} (q_{\lambda'} \restriction \beta_{\bar{i}})$ is not a condition. Hence, if σ is an antichain,

$$c(\lambda, \lambda') = \min\{\bar{i} < i \mid (q_\lambda \restriction \beta_{\bar{i}}) \oplus_{\beta_{\bar{i}}} (q_{\lambda'} \restriction \beta_{\bar{i}}) \notin \mathbb{Q}_{\beta_{\bar{i}}}\}$$

is a well–defined colouring of $[\kappa]^2$. But if H is any homogeneous set with value $\bar{i}$, then $\{q_\lambda \restriction \beta_{\bar{i}} \mid \lambda \in H\}$ is an antichain in $\mathbb{Q}_{\beta_{\bar{i}}}$.

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The case $i = i_0 + 1$ follows easily from earlier cases.

The hardest case is the case $\mathrm{cf}(i) = \kappa$. For this case we use an adaptation of the following key separation argument from Laver–Shelah.

Lemma

(Laver–Shelah) Suppose κ is weakly compact and $(\mathcal{Q}_\beta)_{\beta\leq\tau}$ is a countable support iteration such that $\mathcal{Q}_1 = \mathrm{Col}(\omega_1, {<}\kappa)$ and for all $1 \leq \beta < \tau$, $\mathcal{Q}_{\beta+1} = \mathcal{Q}_\beta * \dot{\mathcal{R}}_\beta$, where $\dot{\mathcal{R}}_\beta$ is the natural forcing for specializing
some given κ–Aronszajn tree $\dot{T}_\beta$. Then $\mathcal{Q}_\beta$ is κ–c.c. for all $\beta \leq \tau$.

Proof sketch: Let $M \preccurlyeq H(\theta)$ containing everything relevant of size κ and such that ${}^{<\kappa}M \subseteq M$ and let $(M_\lambda)_{\lambda<\kappa}$ be a continuous filtration of M. Let $\mathbb{Q}^*_\alpha = \mathbb{Q}_\alpha \cap M$ for all α. By κ–c.c. of $\mathbb{Q}_\alpha$ for alll $\alpha < \tau$ (by induction hypothesis), $\mathbb{Q}^*_\alpha \lessdot \mathbb{Q}_\alpha$ for all $\alpha < \tau$.

Given conditions q^L, q^R, $\alpha \in \mathrm{dom}(f_{q^L}) \cap \mathrm{dom}(f_{q^R})$, $x \in \mathrm{dom}(f_{q^L}(\alpha))$ and $y \in \mathrm{dom}(f_{q^R}(\alpha))$ (x and y may or may not be equal), we say that

- *x and y are separated by $q^L \restriction \alpha$ and $q^R \restriction \alpha$ below λ by means of $\bar{x}$, $\bar{y}$*

if there is $\bar{\rho} < \lambda$, together with $\zeta \neq \zeta'$ in ω_1, such that letting $\bar{x} = (\bar{\rho}, \zeta)$ and $\bar{y} = (\bar{\rho}, \zeta')$,

$$q^L_\lambda \restriction \alpha \Vdash_\alpha \bar{x} <_{\dot{T}_\alpha} x$$

and

$$q^R_\lambda \restriction \alpha \Vdash_\alpha \bar{y} <_{\dot{T}_\alpha} y$$

Let $\sigma = (q_\lambda \mid \lambda < \kappa)$ be a sequence of conditions in $\mathbb{Q}^*_\tau$. Let $\mathcal{F}$ be the weak compactness filter on κ (i.e., $\mathcal{F}$ is the filter generated by the sets $\{\alpha < \kappa \mid (V_\alpha, \in, A \cap V_\alpha) \models \phi\}$, for $A \subseteq V_\kappa$ and for a Π^1_1 sentence ϕ over $(V_\kappa, \in, A)$). $\mathcal{F}$ is a proper normal filter on κ.

Given $X \in \mathcal{F}^+$, say that

$$(q^L_\lambda \mid \lambda \in X), (q^R_\lambda \mid \lambda \in X)$$

is a separating pair for $(q_\lambda \mid \lambda < \kappa)$ if for all $\lambda \in X$:

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Given $X \in \mathcal{F}^+$, say that

$$(q^L_\lambda \mid \lambda \in X), (q^R_\lambda \mid \lambda \in X)$$

is a separating pair for $(q_\lambda \mid \lambda < \kappa)$ if for all $\lambda \in X$:

(1) Both of q^L_λ and q^R_λ extend q_λ.

(2) $\mathrm{dom}(f_{q^L_\lambda}) = \mathrm{dom}(f_{q^R_\lambda})$

(3) For all nonzero $\alpha \in \mathrm{dom}(f_{q^L_\lambda}) \cap M_\lambda$ and all $x \in \mathrm{dom}(f_{q^L_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda}(\alpha)) \setminus (\lambda \times \omega_1)$, x and y are separated below λ at stage α by $q^L_\lambda \restriction \alpha$ and $q^R_\lambda \restriction \alpha$ via some pair $\chi_0(x, y, \alpha, \lambda)$, $\chi_1(x, y, \alpha, \lambda)$.

(4) The following holds for all $\lambda' > \lambda$ in X.
  - (a) $q^L_\lambda \restriction M_\lambda = q^R_{\lambda'} \restriction M_{\lambda'}$
  - (b) $q^L_\lambda \in M_{\lambda'}$

(5) The following holds for all $\lambda' > \lambda$ in X, all nonzero $\alpha \in \mathrm{dom}(q^L_\lambda) \cap \mathrm{dom}(q^R_{\lambda'})$ and all $x \in \mathrm{dom}(f_{q^L_\lambda(\alpha)}) \setminus (\lambda \times \omega_1)$ and $y' \in \mathrm{dom}(f_{q^R_{\lambda'}(\alpha)}) \setminus (\lambda' \times \omega_1)$.
  - (a) $\alpha \in M_\lambda$
  - (b) There are $x' \in \mathrm{dom}(f_{q^L_{\lambda'}(\alpha)}) \setminus (\lambda' \times \omega_1)$ and $y \in \mathrm{dom}(f_{q^R_\lambda(\alpha)}) \setminus (\lambda \times \omega_1)$ such that

$$\chi_0(x, y, \alpha, \lambda) = \chi_0(x', y', \alpha, \lambda')$$

and

$$\chi_1(x, y, \alpha, \lambda) = \chi_1(x', y', \alpha, \lambda')$$

The following claim is easy.

Claim

Let $X \in \mathcal{S}$ and suppose $\sigma^L = (q^L_\lambda \mid \lambda \in X)$, $\sigma^R = (q^R_\lambda \mid \lambda \in X)$ is a separating pair for σ. Then for all $\lambda < \lambda'$ in X,

$$q^L_\lambda$$

and

$$q^R_{\lambda'}$$

are compatible conditions.

Hence, it suffices to prove that there is $\sigma^L = (q^L_\lambda \mid \lambda \in X)$, $\sigma^R = (q^R_\lambda \mid \lambda \in X)$, a separating pair for σ. But this follows essentially from a construction in ω steps such that

- at every step we separate some given sequence of pair of nodes x, y,

followed by a pressing–down argument using the normality of $\mathcal{F}$.

The relevant separation, at every step of the construction, is effected via a Π^1_1 reflection argument: There is a measure 1 set C in $\mathcal{F}$ of $\lambda < \kappa$ such that, for relevant α,

- $M_\lambda \cap \mathbb{Q}_\alpha \lessdot \mathbb{Q}_\alpha$ and
- $M_\lambda \cap \mathbb{Q}_\alpha$ forces, over V, that $\dot{T}_\alpha \cap M_\lambda$ has no λ–branches.

Using this idea one can find suitable conditions

$$q_\lambda^{LL} \leq q_\lambda^{L}$$

and

$$q_\lambda^{RR} \leq q_\lambda^{R}$$

such that

- $q_\lambda^{LL} \restriction M_\lambda = q_\lambda^{RR} \restriction M_\lambda$ and
- forcing conflicting information regarding the projections of x and y to some level below λ

(if this were not possible, we would be able to find λ–branches through $\dot{T}_\alpha \cap M_\lambda$ in the $M_\lambda \cap \mathbb{Q}_\alpha$–extension, which is impossible). $\square$

Thank you!