Borel complexity in hyperspaces up to equivalence

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The equivalence

We say that classes of topological spaces $\mathcal{C}$, $\mathcal{D}$ are *equivalent* if every member of $\mathcal{C}$ is homeomorphic to a member of $\mathcal{D}$ and vice versa. We write $\mathcal{C} \cong \mathcal{D}$ and we put

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- Namely, there is the *saturated family*

$$\max([\mathcal{C}]) = \bigcup_{X \in \mathcal{C}} \{Y \in \mathcal{K}([0,1]^\omega) : X \cong Y\}.$$

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- Our motivation lies in *compactifiable classes*.

Compactifiable and Polishable classes

Definition

A class of topological spaces $\mathcal{C}$ is called

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We define the strong variants because of their direct connection with hyperspaces.

Relation to hyperspaces

Theorem

The following conditions are equivalent for a class of spaces $\mathcal{C}$.

1. $\mathcal{C}$ is strongly compactifiable.
2. There is a metrizable compactum X and a closed family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
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1. $\mathcal{C}$ is a strongly Polishable class of compacta.
2. There is a Polish space X and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
3. There is a G_δ family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.
4. There is a closed family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

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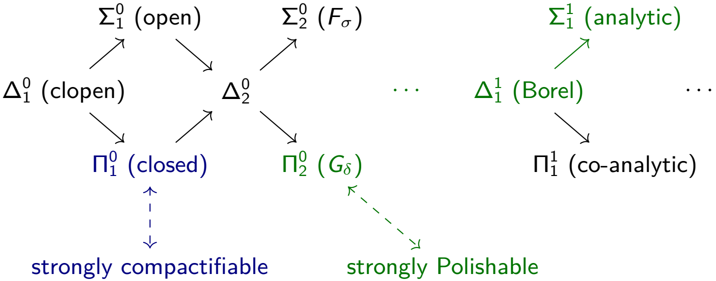

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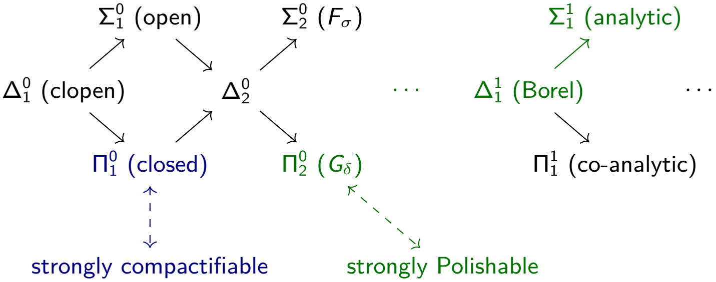

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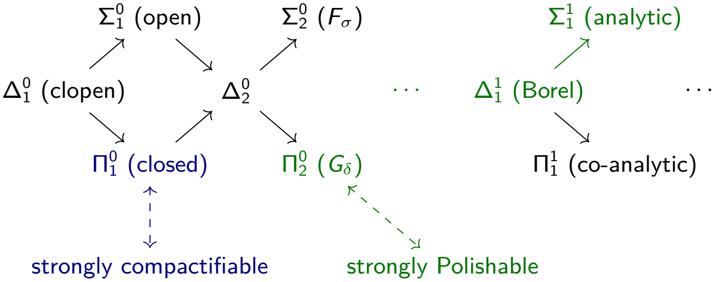

What about clopen, open, and F_σ subsets of $\mathcal{K}([0,1]^\omega)$?

Proposition

There are only four clopen subsets of $\mathcal{K}([0,1]^\omega)$:

$$\emptyset, \ \ \{\emptyset\}, \ \ \mathcal{K}([0,1]^\omega) \setminus \{\emptyset\}, \ \ \mathcal{K}([0,1]^\omega).$$

Principal upper classes

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- We define a partial order $\leq$ on $T \cup \{\infty\}$:
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- For $t \in T \cup \{\infty\}$ we define the *principal upper class* $\mathcal{U}_t := \{X : t(X) \geq t\}$.

Principal upper classes

Examples

We have the following classes of metrizable compact spaces:

- $\mathcal{U}_{0,0} = \{\emptyset\}$,
- $\mathcal{U}_{1,0}$ – all nonempty compacta,
- $\mathcal{U}_{1,1}$ – all infinite compacta,
- $\mathcal{U}_{2,0} \cup \mathcal{U}_{1,1}$ – all nondegenerate compacta,
- $\mathcal{U}_{m,0}$ – all compacta with at least m components,
- $\mathcal{U}_{m,0} \cup \mathcal{U}_{1,1}$ – all compacta with at least m points.

Nice antichains

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Let $R \subseteq T \cup \{\infty\}$.

- We say that R is *nice* if $(m, 0) \in R$ for some $m > 0$ whenever $R \cap (T_+ \cup \{\infty\}) \neq \emptyset$. This holds if and only if $\bigcup_{t \in R} \mathcal{U}_t$ contains a nonempty finite space whenever it contains a nonempty space.

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- We say that R is an *antichain* if it is pairwise $\leq$-incomparable. Note that every antichain is finite, and that no nice antichain contains ∞.
- By $A(R)$ we denote the set of all $\leq$-minimal elements of R. Note that this is the only antichain A such that $\bigcup_{t \in A} \mathcal{U}_t = \bigcup_{t \in R} \mathcal{U}_t$. It follows that $A(R)$ is nice if and only if R is nice.

Open classes – the outline

Part 1

The set $\bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open if and only if R is nice.

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The set $\bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open if and only if R is nice.

Part 2

Every open set $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ is equivalent to $\bigcup_{X \in \mathcal{U}} \mathcal{U}_{t(X)}$.

Special open classes

Let $s\colon I \to \mathbb{N}_+$ be a finite function.

- The *special open class* $\mathcal{O}_s$ is the class of all metrizable compacta K having a clopen decomposition $\{K_i : i \in I\}$ such that $|K_i| \geq s(i)$ for every $i \in I$.

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Let X be a metrizable space and let $\mathcal{U} \subseteq \mathcal{K}(X)$ be open.

- The set $\mathcal{U}$ is *of the shape s* if there are disjoint open sets $U_i \subseteq X$ and $V_{i,j} \subseteq U_i$ for $i \in I$ and $j < s(i)$ such that
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- The set $\mathcal{U}$ is *exactly of the shape* s if additionally every set $U_i^+ \cap \bigcap_{j<s(i)} V_{i,j}^-$ contains a connected space.

Observation

A space $K \in \mathcal{K}(X)$ has a neighborhood of the shape s in $\mathcal{K}(X)$ if and only if $K \in \mathcal{O}_s$. It follows that $\mathcal{O}_s \cap \mathcal{K}(X)$ is open.

Special open classes

Proposition

For every finite function $s\colon I \to \mathbb{N}_+$ there is a nice antichain $R_s \subseteq T$ such that $\mathcal{O}_s = \bigcup_{t \in R_s} \mathcal{U}_t$.

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Examples

- $\mathcal{O}_\emptyset = \mathcal{U}_{0,0} = \{\emptyset\}$.
- $\mathcal{O}_{(1)} = \mathcal{U}_{1,0}$ – all nonempty compacta.
- $\mathcal{O}_{(2)} = \mathcal{U}_{1,1} \cup \mathcal{U}_{2,0}$ – all nondegenerate compacta.
- $\mathcal{O}_{(m)} = \mathcal{U}_{1,1} \cup \mathcal{U}_{m,0}$ – all compacta with at least m points.
- $\mathcal{O}_{(1\colon i<m)} = \mathcal{U}_{m,0}$ – all compacta with at least m components.
- $\mathcal{O}_{(1,1,1,2,3,4)} = \mathcal{U}_{6,3} \cup \mathcal{U}_{7,2} \cup \mathcal{U}_{9,1} \cup \mathcal{U}_{12,0}$.

Open classes – Part 1

To every type $t \in T \cup \{\infty\}$ we associate a set of finite functions

$$S_t := \begin{cases} \{s\colon m \to \mathbb{N}_+ : |\{i < m : s(i) > 1\}| \leq n\} & \text{if } t = (m, n), \\ \{s\colon m \to \mathbb{N}_+ : m > 0\} & \text{if } t = \infty. \end{cases}$$

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Proposition

Let $R \subseteq T \cup \{\infty\}$.

The set $\bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open if and only if R is nice.

Open classes – Part 2

Proposition

Let $t \in T \cup \{\infty\}$ and let M be metrizable. Every $X \in \mathcal{U}_t \cap \mathcal{K}(M)$ has a neighborhood basis such that for every basic set $\mathcal{U}$ there is $s \in S_t$ such that $\mathcal{U}$ is exactly of the shape s.

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Proposition

Let $s\colon I \to \mathbb{N}_+$ be a finite function. For every compactum $Y \in \mathcal{O}_s$ and every open set $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ exactly of the shape s there is a compactum $Y' \in \mathcal{U}$ homeomorphic to Y.

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- For every $R \in \mathcal{R}$ we have $\mathcal{O}_R \cong \mathcal{O}_R \cap \mathcal{K}([0,1]^\omega)$, which is open.

F_σ classes

Proposition

Let us have

- a finite function $s\colon I \to \mathbb{N}_+$ and a metrizable space X,
- an open set $\mathcal{U} \subseteq \mathcal{K}(X)$ of the shape s,
- an open set $\mathcal{V} \subseteq \mathcal{K}([0,1]^\omega)$ exactly of the shape s.

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- an open set $\mathcal{U} \subseteq \mathcal{K}(X)$ of the shape s,
- an open set $\mathcal{V} \subseteq \mathcal{K}([0,1]^\omega)$ exactly of the shape s.

For every compact set $\mathcal{H} \subseteq \mathcal{U}$ there is a compact set $\mathcal{F} \subseteq \mathcal{V}$ and a homeomorphism $\Phi\colon \mathcal{H} \to \mathcal{F}$ such that $\Phi(H) \cong H$ for every $H \in \mathcal{H}$.

F_σ classes

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For every strongly compactifiable class $\mathcal{C} \subseteq \mathcal{O}_s$ there is a compact zero-dimensional family $\mathcal{F} \subseteq \mathcal{V}$ equivalent to $\mathcal{C}$

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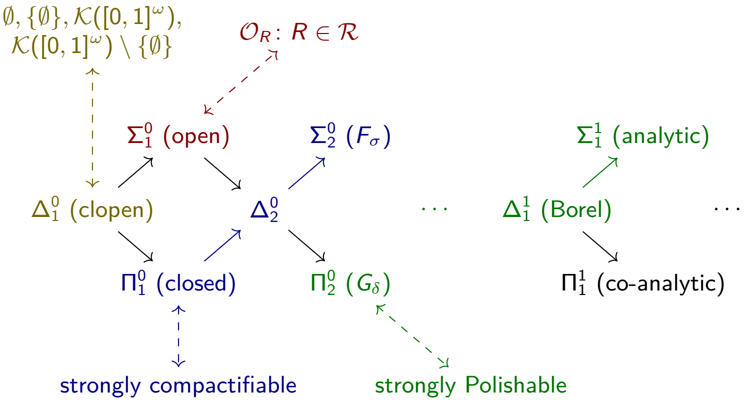
Theorem

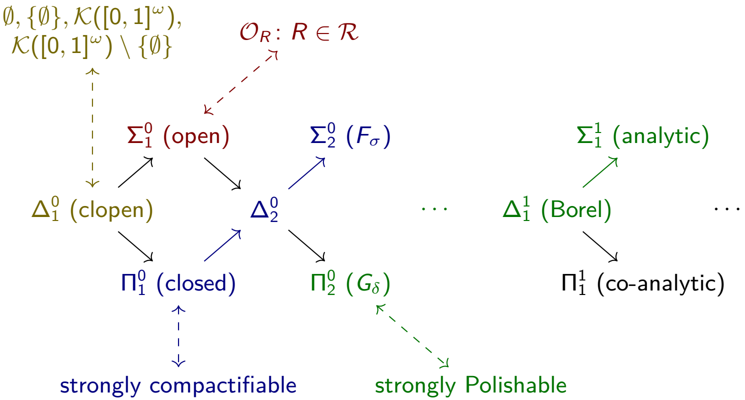
Every F_σ subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed subset.

Strongly compactifiable classes are stable under countable unions.

Conclusion


$\emptyset, \{\emptyset\}, \mathcal{K}([0,1]^\omega),$
$\mathcal{K}([0,1]^\omega) \setminus \{\emptyset\}$
$\mathcal{O}_R \colon R \in \mathcal{R}$
Σ^0_1 (open)
Σ^0_2 (F_σ)
Σ^1_1 (analytic)
Δ^0_1 (clopen)
Δ^0_2
$\cdots$
Δ^1_1 (Borel)
$\cdots$
Π^0_1 (closed)
Π^0_2 (G_δ)
Π^1_1 (co-analytic)
strongly compactifiable
strongly Polishable

Conclusion



Thank you for your attention.