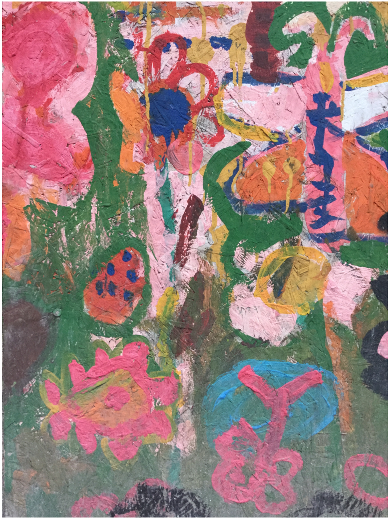
Walks of higher order

Jeffrey Bergfalk

Novi Sad
July 2018

So wrong it's right!
Overview
Walks
and cohomology
Higher coherence
Higher walks
Conclusion

This week's made me happy.



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Two foundational theorems

Fix natural numbers $m \neq n$.

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In other words: foundational theorems in set theory and algebraic topology, respectively, signal utterly opposite approaches to the concept of dimension.

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Oil and water

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Theorem

Let f be

- *an order-preserving map from* ω_1 *to* $\mathbb{R}$*, or*
- *an order-preserving map from* $\mathbb{R}$ *to* ω_1*, or*
- *a continuous map from* ω_1 *to* $\mathbb{R}$*, or*
- *a continuous map from* $\mathbb{R}$ *to* ω_1*.*

Then f is eventually constant.

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Doing it wrong

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The quote's from *Elementary Principles of Topology*, which came my way when I was turning to something even wronger, probably, than anything Alexandroff had had in mind:

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The cohomology of the ordinals

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\check{\mathrm{H}}^3$	**0**	**0**	**0**	**nonzero**
$\check{\mathrm{H}}^2$	**0**	**0**	**nonzero**	consistently nonzero
$\check{\mathrm{H}}^1$	**0**	**nonzero**	*independent*	*independent*
$\check{\mathrm{H}}^0$	**nonzero**	**nonzero**	**nonzero**	**nonzero**
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3. Pictured, plainly, are phenomena of *dimension*.

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3. higher Čech cohomology and higher nontrivial coherence, and
4. higher-order walks.

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Walks

Walks on the countable ordinals

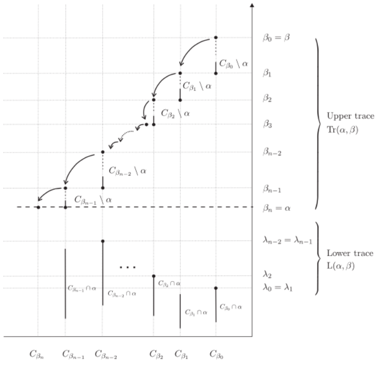
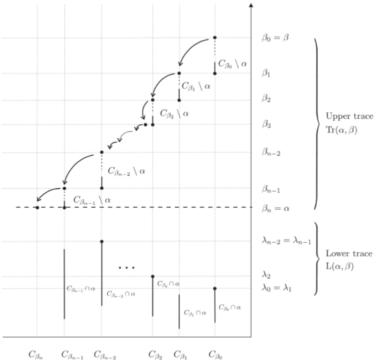


Figure 2.1: The walk and its traces.

Walks on the countable ordinals



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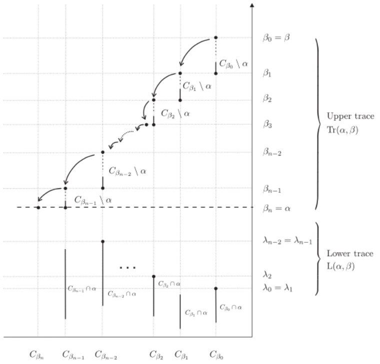

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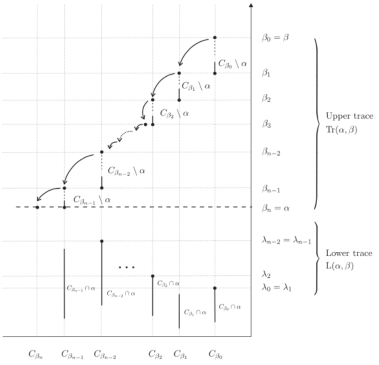

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(Nonconstructive) input: a *C-sequence* $\langle C_\alpha \mid \alpha \in \omega_1 \rangle$. Here each C_α is a (minimal-ordertype) witness to the cofinality of α.

(Recursive) output: a finite *walk*, recorded as $\mathrm{Tr}(\alpha, \beta)$, for any $\alpha < \beta < \omega_1$.

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An *all* and an *only*

[Walks,] despite [their] simplicity, can be used to derive virtually ***all*** *known other structures that have been defined so far on* ω_1*.*

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An interesting phenomenon that one realizes while analyzing walks on ordinals is the special role of the first uncountable ordinal ω_1 *in this theory. [...] The first uncountable cardinal is the* ***only*** *cardinal on which the theory can be carried out without relying on additional axioms of set theory.*

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(Why should these facts be so?)

rho functions

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These "derivations" are largely by way of the rho functions; we foreground two:

- $\rho_2(\alpha, \beta) = |\mathrm{Tr}(\alpha, \beta)|$ ("width")
- $\rho_1(\alpha, \beta) = \max\{|C_\xi \cap \alpha| : \xi \in \mathrm{Tr}(\alpha, \beta)\}$ ("height")

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while there exists no $\tilde{\rho}_1 : \omega_1 \to \mathbb{N}$ such that

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ρ_2, also, satisfies (1) but not (2), if we read $=^*$ as *equality modulo locally constant functions*.

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Example

The functor $\mathcal{A}_d = U \mapsto \{f : U \to A \,|\, f \text{ is locally constant}\}$ is a (pre)sheaf.

Čech cohomology

Definition (Part 1)

Fix $\mathcal{V} = \{V_\alpha \mid \alpha \in \delta\}$, an open cover of X. Write $\mathrm{H}^n(\mathcal{V}, \mathcal{P})$ for the n^{th} cohomology group of the cochain complex

$$\to L^0(\mathcal{V}, \mathcal{P}) \to \cdots \to L^j(\mathcal{V}, \mathcal{P}) \xrightarrow{d^j} L^{j+1}(\mathcal{V}, \mathcal{P}) \to \ldots$$

Here

$$L^j(\mathcal{V}, \mathcal{P}) = \prod_{\vec{\alpha} \in [\delta]^{j+1}} \mathcal{P}(V_{\vec{\alpha}}),$$

where $V_{\vec{\alpha}} = V_{\alpha_0} \cap \cdots \cap V_{\alpha_{j-1}}$. Write then $p_{\vec{\alpha}\vec{\beta}}$ for $p_{V_{\vec{\alpha}} V_{\vec{\beta}}}$, and define $d^j : L^j(\mathcal{V}, \mathcal{P}) \to L^{j+1}(\mathcal{V}, \mathcal{P})$ by

$$d^j f \;:\; \vec{\alpha} \mapsto \sum_{i=0}^{j+1} (-1)^i p_{\vec{\alpha}^i \vec{\alpha}}(f(\vec{\alpha}^i))$$

Čech cohomology

Definition (Part 2)

Write $\mathcal{V} \leq \mathcal{W}$ if the open cover $\mathcal{W}$ refines $\mathcal{V}$, i.e., if there exists some $r_{\mathcal{W}\mathcal{V}} : \mathcal{W} \to \mathcal{V}$ such that $W \subseteq r_{\mathcal{W}\mathcal{V}}(W)$ for each $W \in \mathcal{W}$. The induced $r^*_{\mathcal{W}\mathcal{V}} : \mathrm{H}^n(\mathcal{V}, \mathcal{P}) \to \mathrm{H}^n(\mathcal{W}, \mathcal{P})$ is independent of the choice of refining map $r_{\mathcal{W}\mathcal{V}}$. Hence these $r^*_{\mathcal{W}\mathcal{V}}$ $(\mathcal{V} \leq \mathcal{W})$ define, in turn, a direct limit

$$\check{\mathrm{H}}^n(X, \mathcal{P}) := \varinjlim_{\mathcal{V} \in \mathrm{Cov}(X)} \mathrm{H}^n(\mathcal{V}, \mathcal{P}) \tag{3}$$

This limit is the n^{th} *Čech cohomology group of* X, *with respect to the presheaf* $\mathcal{P}$.

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$$f : (\alpha,\beta) \mapsto \rho_1(\,\cdot\,,\beta)\restriction_\alpha - \rho_1(\,\cdot\,,\alpha) \text{ fits the bill.}$$

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A computation

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A computation

$[f] \in \mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_\mathbb{Z})$ is zero iff there exists some g with

$$g(\alpha) : \alpha \to \oplus_\alpha \mathbb{Z}$$

such that

$$g(\beta)\restriction_\alpha - g(\alpha) = f(\alpha, \beta) \text{ for all } \alpha \leq \beta < \omega_1$$

This in our case would entail that

$$(\rho_1(\,\cdot\,, \beta) - g(\beta))\restriction_\alpha = \rho_1(\,\cdot\,, \alpha) - g(\alpha)$$

Write then $\tilde{\rho}_1$ for

$$\varinjlim_{\alpha \in \omega_1} \rho_1(\,\cdot\,, \alpha) - g(\alpha)$$

$\tilde{\rho}_1$ is then a function $\omega_1 \to \mathbb{Z}$ differing from each $\rho_1(\,\cdot\,\alpha)$ by $g(\alpha)$, i.e., on only finitely many coordinates – but there is no such function. Hence $0 \neq [f] \in \mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_\mathbb{Z})$.

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Theorem

$\mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_A)$ *is the group of coherent families of functions* $\{\varphi_\beta : \beta \to A \mid \beta \in \omega_1\}$, *quotiented by the group of trivial families of functions* $\{\varphi_\beta : \beta \to A \mid \beta \in \omega_1\}$.

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$$\mathrm{H}^1(\mathcal{U}_{\omega_1}, \mathcal{D}_A) \cong \check{\mathrm{H}}^1(\omega_1, \mathcal{D}_A) \cong \check{\mathrm{H}}^1(\omega_1, \mathcal{A}_d)$$

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for all natural numbers k *and* n*.*

n-coherence

Definition

For $n \in \mathbb{N}$, a family $\Phi_n = \{\varphi_{\vec{\alpha}} : \alpha_0 \to A \,|\, \vec{\alpha} \in [\varepsilon]^n\}$ is *n-coherent* if

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For $n > 1$, Φ_n is *n-trivial* if there exists a $\Psi_{n-1} = \{\psi_{\vec{\alpha}} : \alpha_0 \to A \,|\, \vec{\alpha} \in [\varepsilon]^{n-1}\}$ such that

$$\sum_{i=0}^{n-1} (\text{-}1)^i \psi_{\vec{\alpha}^i} =^* \varphi_{\vec{\alpha}}$$

for all $\vec{\alpha} \in [\varepsilon]^n$.

A vanishing theorem

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See the theorem as the zeros of the chart from before:

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(again)

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\check{H}^3$	**0**	**0**	**0**	**nonzero**
$\check{H}^2$	**0**	**0**	**nonzero**	consistently nonzero
$\check{H}^1$	**0**	**nonzero**	*independent*	*independent*
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	ω	ω_1	ω_2	ω_3

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The natural next question is whether $\check{\mathrm{H}}^n(\omega_n, \mathcal{A}_d)$ vanishes.

And if ω_1 is any guide, this is really a question of "higher order walks."

Higher walks

A hint

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This work has pointed insistently to the following sorts of structures:

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Fix a C-sequence $\langle C_\gamma \mid \gamma < \omega_2 \rangle$.

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$$C_{\beta\gamma} := \pi^{-1}(C_{\mathrm{otp}(C_\gamma \cap \beta)})$$

where π is the order-isomorphism $C_\gamma \cap \beta \to \mathrm{otp}(C_\gamma \cap \beta)$.

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In particular, the three-coordinate $\mathrm{Tr}^2(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ implicit in the previous slide exhibits the sort of coherence relations that $\mathrm{Tr}(\,\cdot\,,\alpha)$ and $\mathrm{Tr}(\,\cdot\,,\beta)$ do in the classical case.

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And all that I'm describing extends naturally to any finite *n*.

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$$TR^2(\pm, \sigma, \alpha, \beta, \gamma) =$$

<u>Case</u>: $\beta \in C_\gamma$:

$$\begin{aligned} &\{\, (\mp, \sigma, \min(C_{\beta\gamma} \backslash \alpha))\,\} \\ &\cup\ TR^2(\pm, \sigma^\frown 0, \alpha, \min(C_{\beta\gamma}(\alpha)), \gamma) \\ &\cup\ TR^2(\mp, \sigma^\frown 1, \alpha, \min(C_{\beta\gamma}(\alpha)), \beta) \end{aligned}$$

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In that case, though, it was pointless to record the *constant* data of sign ($\pm$), while the choice-of-step data (σ) appeared simply as an index ($|\sigma| = i$ for $\beta_i \in \mathrm{Tr}(\alpha, \beta)$). (Compare how, in more geometric contexts, *orientation* only assumes its full importance in dimensions greater than two.) The $n = 1$ case of the following, then, is the classical ρ_2:

$$\rho_2^n(\vec{\alpha}) := \mathrm{neg}(TR^n(\vec{\alpha})) - \mathrm{pos}(TR^n(\vec{\alpha}))$$

where *neg* and *pos* simply count the number of negative and positive terms, respectively, in $TR^n(\vec{\alpha})$.

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Theorem

$\check{\mathrm{H}}^n(\omega_n, \mathcal{D}_A) \neq 0$, for $A = \bigoplus_{\omega_n} \mathbb{Z}$, for all $n \geq 0$.

If we expand our assumptions

Theorem (B., Lambie-Hanson)

Suppose $V = L$, and $n \geq 1$, and $\kappa \geq \aleph_n$ is a regular cardinal that is not weakly compact.

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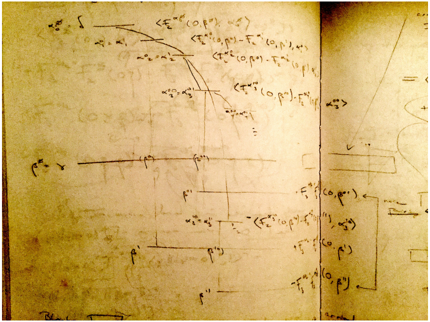
Theorem (Todorcevic)

Assume the P-Ideal Dichotomy, and let A be an abelian group. Then $\check{\mathrm{H}}^1(\varepsilon, \mathcal{A}_d) \neq 0$ if and only if the cofinality of ε is ω_1.

What I'm saying

What I want most essentially to tell you — particularly those of you thinking of triangles, or ω_2, or colorings, particularly *in ZFC* — is that these $C_{\beta\gamma}$-structures are extraordinarily productive and rich.



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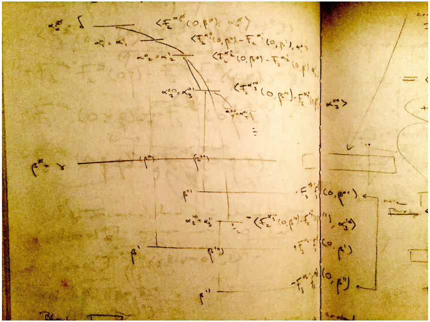

I encourage people to play with them. I'm interested in whatever you find. I do *not* worry that any of us will exhaust their possibilities.

Last thoughts and suggestions

This body of research might be viewed as addressing most fundamentally the question

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They each have their own lives...)

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For this *is* a situation calling ultimately either for remedy or for explanation.

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This body of research might be viewed as addressing most fundamentally the question

Why can we say so much and so little, respectively, about the ZFC combinatorics of ω_1 and of higher ω_n?

(Guiding, for me, has been a statement of Todorcevic's:

They each have their own lives...)

For this *is* a situation calling ultimately either for remedy or for explanation. And emergent in an approach centered on dimension are compelling generalizations of the ω_1 case, namely

rich and distinctive ZFC combinatorics fundamentally expressive of the topology of ω_n, for each $n \in \mathbb{N}$.

So wrong it's right!
Overview
Walks
and cohomology
Higher coherence
Higher walks
Conclusion

Thanks