Killing P-points made simple

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P-points

Definition

An ultrafilter $\mathcal{U}$ on ω is a *P-point* if for every $\{ U_i \mid i \in \omega \} \subset \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $U \setminus U_i$ is finite for each $i \in \omega$.

P-points do exist if

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Non-existence

(1977) There is a generic extension with no P-points (Shelah).

Killing P-points

Shelah's Proposition

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- The inequality $\operatorname{cof}\mathcal{N} < \operatorname{cov} v_0$ implies that there are no P-points.

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Let $\mathcal{U}$ be a non-principal ultrafilter.
$\mathbf{PS}^\omega$ *forces that $\mathcal{U}$ cannot be extended to a P-point.*

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Find sequence of finite partitions of ω.
$\langle \mathscr{D}_n = \langle D_n^0, D_n^1, \dots, D_n^{\ell(n)} \rangle \restriction n \in \omega \rangle$
such that for each selector $c \colon \omega \to \omega$
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How to show that each pseudo-intersection of a sequence $\langle D_n \mid n \in \omega \rangle$ is in $\mathcal{U}^*$?
For each $f\colon \omega \to \omega$ find an increasing sequence $\langle b_n \in \omega \mid n \in \omega \rangle$ such that $f(n) < b_n$ and $\bigcup_{n\in\omega} (D_n \cap [b_n, b_{n+1})) \in \mathcal{U}^*$.

The conditions of the *Silver forcing* $\mathbf{PS}$ are partial functions $p; \omega \to 2$ with co-infinite domain, the order is reversed inclusion.

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Fact

The forcing $\mathbf{PS}^\omega$ is proper and has the Sacks property.

I.e. for every $p \in \mathbf{PS}$ and for every $\dot{c}$ such that $p \Vdash \dot{c} \in \omega^\omega$ there is $q \leq p$ and $\{ C_n \in [\omega]^{n+1} \mid n \in \omega \}$ such that $q \Vdash \dot{c}(n) \in C_n$.

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Lemma

If $m > n^2$, then for each $C \in [m]^n$ there exists $s \in m$ such that $C \cap (C - \{ s \}) = \emptyset \mod m$.

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Fact

Let $k < n \neq 1$, and r be a Silver generic real over V.
Then $D^k_n(r)$ is an independent real over V.
I.e. $D^k_n(r) \cap A \neq \emptyset$ for each $A \in [\omega]^\omega \cap V$.