Set theory today

A conference in honor of Georg Cantor

Omer Ben Neria · Jörg Brendle · David Chodounsky · James Cummings · Mirna Dzamonja · Oswaldo Guzman · Radek Honzik · Yurii Khomskii · Paul Larson · Diego Mejia · Julien Melleray · Heike Mildenberger · Luca Motto Ros · Grigor Sargsyan · Asger Törnquist · Todor Tsankov · Matteo Viale · Jindrich Zapletal

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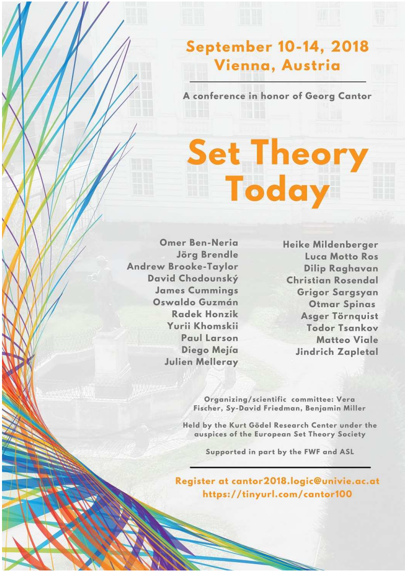

ESTC2019 $\bigoplus$ Advanced Class

Advanced Class – last week of June 2019, Vienna

6 Tutorials, four one hour lecture by Justin Moore, Jörg Brendle, Slawomir Solecki, Alexander Kechris, Hugh Woodin and Matteo Viale.

6-9 Thematic Discussion Sessions

ESTC2019 – first week of July 2019, Vienna

Invited speakers include Moti Gitik, Maryanthe Malliaris, Mirna Dzamonja, Boaz Tsaban, Piotr Koszmider, Justin Moore, Joerg Brendle, Slawomir Solecki, Alexander Kechris and Matteo Viale.

Ladder system uniformization on trees

Dániel T. Soukup

http://www.logic.univie.ac.at/∼soukupd73/

Supported in part by **FWF Grant I1921, OTKA 113047**.

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An ω_1-uniformization of $\mathbf{f}$ is a map $\varphi : \omega_1 \to \omega$ so that for any $\alpha \in \lim(\omega_1)$, $\varphi(\xi) = f_\alpha(\xi)$ **for almost all** $\xi \in C_\alpha$.

Possible restrictions: only 2-colourings, or

each f_α is constant (monochromatic colouring).

Introduction to uniformization

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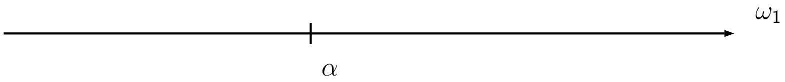

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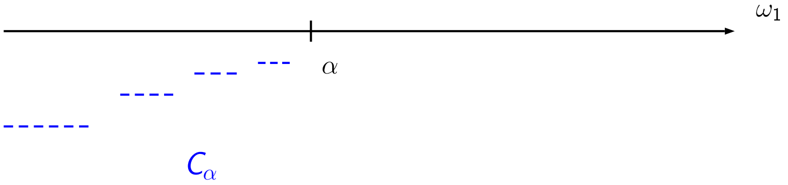

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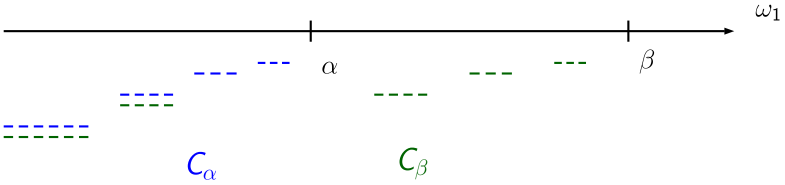

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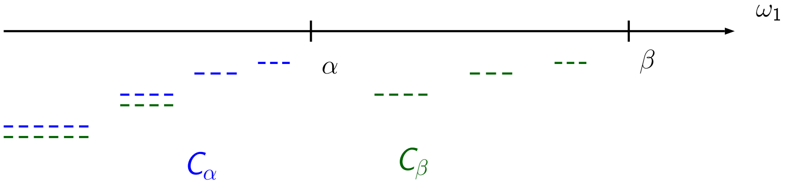

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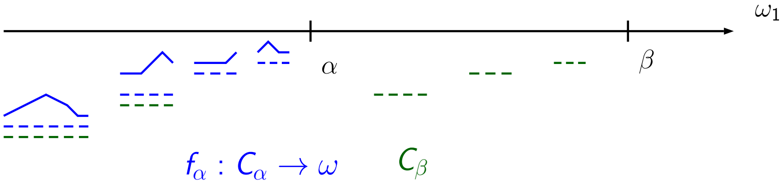

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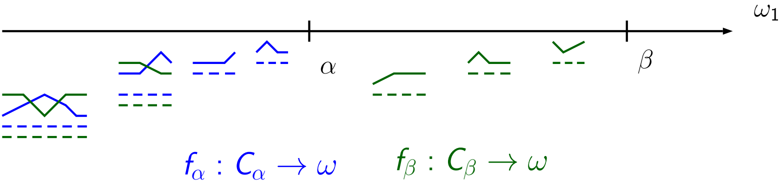

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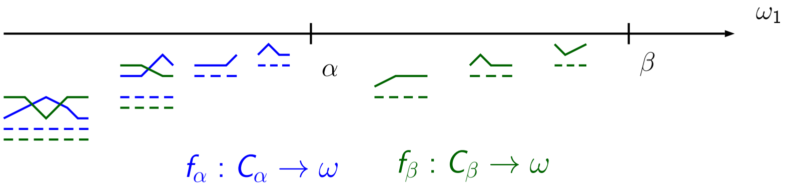

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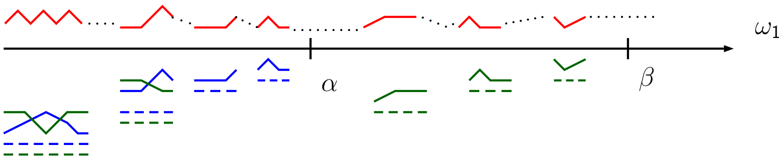

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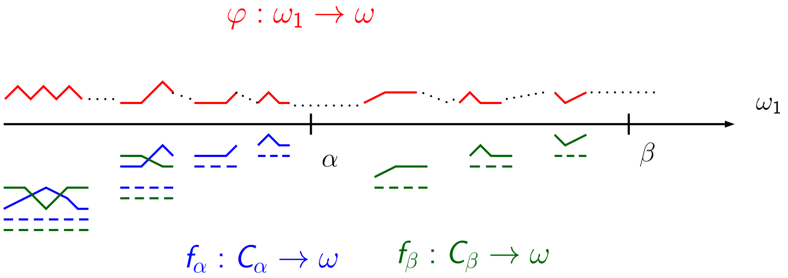

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When do uniformizations exist?

$MA_{\aleph_1}$ implies that any ladder system colouring has an ω_1-uniformization.

$2^{\aleph_0} < 2^{\aleph_1}$ implies that any ladder system has a monochromatic 2-colouring without ω_1-uniformization.

The motivation to study these objects come from the Whitehead-and related algebraic problems, various topological questions (e.g. normal Moore-space conjecture), the study of forcing axioms that allow CH.

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A few words on trees

We will look at pruned trees T of height ω_1.

- ω_1 itself is a tree of height ω_1;
- **Aronszajn-trees**: all the levels and chains are countable;
- **Suslin-trees**: all the antichains and chains are countable;
- $\bar{\sigma}\mathbb{Q}$, the set of all **well ordered $t \subset \mathbb{Q}$ which have a maximum** with the initial segment relation;
  - its levels have size continuum, but chains are countable;
  - special: the union of countably many antichains.

We say $S \subseteq T$ is a **subtree** if S is downward closed and pruned.

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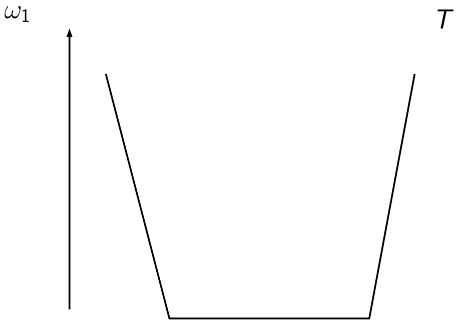

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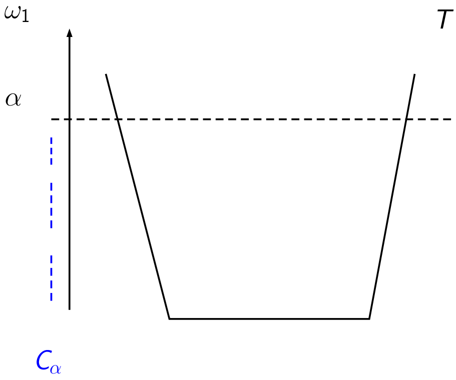

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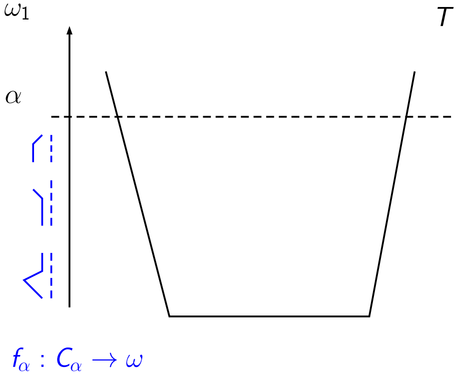

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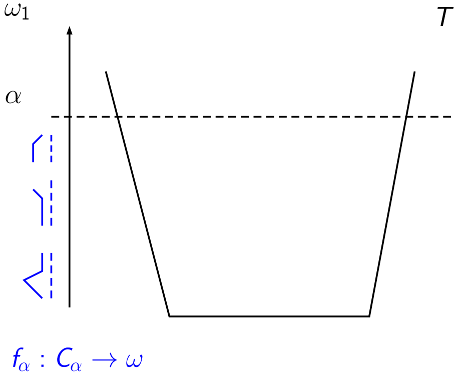

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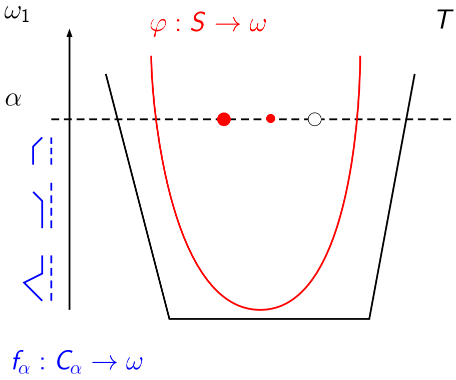

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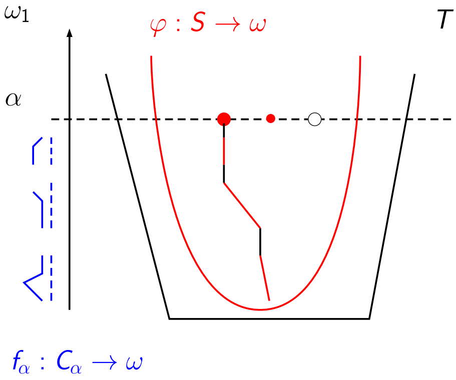

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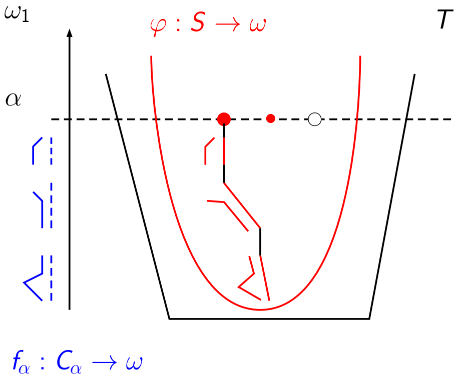

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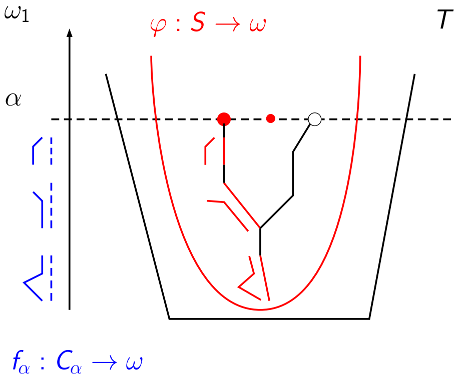

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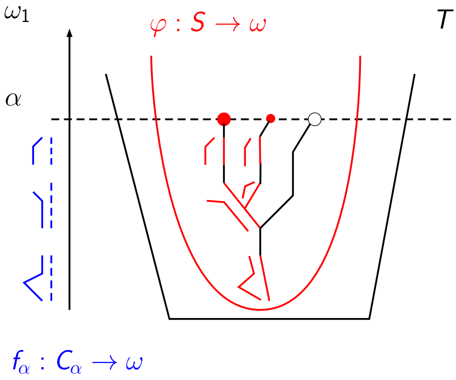

Uniformization and minimal linear orders

A linear order L is minimal if L embeds into all its suborders of size $|L|$.

- the only minimal linear orders of size $\aleph_0$ are $\pm\omega$;
- under PFA, **Baumgartner:** any $\aleph_1$-dense set of reals is minimal and **Todorcevic:** there are minimal A-lines;
- **Baumgartner:** under $\diamondsuit^+$, there are minimal A-lines.

Consistently, the only minimal linear orders of size $\aleph_1$ are $\pm\omega_1$.

Consistently, CH holds and for any A-tree T, any ladder system colouring has a T-uniformization.

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- under PFA, **Baumgartner:** any $\aleph_1$-dense set of reals is minimal and **Todorcevic:** there are minimal A-lines;
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Consistently, the only minimal linear orders of size $\aleph_1$ are $\pm\omega_1$.

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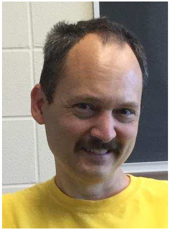

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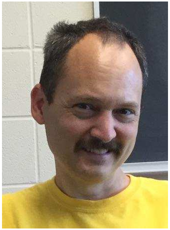

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Under $\diamondsuit^+$, there is a lexicographically ordered Suslin-tree which is a minimal linear order.

Baumgartner 1982/2017

Question: does a single Suslin-tree or $\diamondsuit$ suffice for the construction?

Consistently, there is a Suslin-tree and the only minimal linear orders of size $\aleph_1$ are $\pm\omega_1$.

DTS 2018

- $(V = L)$ take a full Suslin-tree R,
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Non-uniformization results on trees

$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow$ Any l.s. **C** has a monochrom. 2-colouring without ω_1-uniformization.

$2^{\aleph_0} < 2^{\aleph_1} \Rightarrow$ for **any Suslin-tree T and any C**, there is a monochrom. 2-colouring without T-uniformization.

$\diamondsuit \Rightarrow$ **any A-tree T there is a C** and a monochrom. 2-colouring without T-uniformization.

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$\diamondsuit \Rightarrow$ any A-tree T and l.s. **C** there is a 2-colouring without T-uniformization.

$\diamondsuit^+$ implies that for any ladder system **C**, there is an A-tree T so that any monochromatic colouring of **C** has a T-uniformization.

Without any extra assumptions (in ZFC):

There is a map $\varphi : \bar{\sigma}\mathbb{Q} \to \omega$ so that for any ladder system colouring **f** there is an A-tree $T \subseteq \bar{\sigma}\mathbb{Q}$ so that $\varphi \upharpoonright T$ is a T-uniformization of **f**.

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DTS 2018

Thank you very much! Any questions?

James E. Baumgartner

March 23, 1943 – December 28, 2011

For more details and open problems:

D. T. Soukup, **Ladder system uniformization on trees I & II**, prerint, **arXiv: 1806.03867**

D. T. Soukup, **A model with Suslin trees but no minimal uncountable linear orders other than ω_1 and $-\omega_1$**, submitted to the Israel Journal of Math., **arXiv: 1803.03583**.