Global Chang's Conjecture and singular cardinals

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Introduction

Theorem (Löwenheim-Skolem)

Let $\mathfrak{A}$ be an infinite model in a countable first-order language. For every infinite cardinal $\kappa \leq |\mathfrak{A}|$, there is an elementary $\mathfrak{B} \prec \mathfrak{A}$ of size κ.

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Generalizing this, $(\kappa_1, \kappa_0) \twoheadrightarrow (\mu_1, \mu_0)$ says that for every structure $\mathfrak{A}$ on κ_1 in a countable language, there is a substructure $\mathfrak{B}$ of size μ_1 such that $|\mathfrak{B} \cap \kappa_0| = \mu_0$.

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If $\kappa_1 = \kappa_0^+$ and $\mu_1 = \mu_0^+$, this is equivalent to an analogue of Löwenheim-Skolem for a logic between first and second order. This logic includes a quantifier Qx, where $Qx\varphi(x)$ is valid when the number of x's satisfying $\varphi(x)$ is equal to the size of the model.

Lemma

Suppose $\kappa, \lambda \leq \delta$ *and* $\kappa^\lambda \geq \delta$*. Then there is a structure* $\mathfrak{A}$ *on* δ *such that for every* $\mathfrak{B} \prec \mathfrak{A}$*,*

$$|\mathfrak{B} \cap \kappa|^{|\mathfrak{B} \cap \lambda|} \geq |\mathfrak{B} \cap \delta|.$$

Corollary

If $(\kappa_1, \kappa_0) \twoheadrightarrow (\mu_1, \mu_0)$*,* $\nu \leq \kappa_0$*, and* $\kappa_0^\nu \geq \kappa_1$*, then* $\mu_0^{\min(\mu_0, \nu)} \geq \mu_1$*.*

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Corollary

If $(\kappa_1, \kappa_0) \twoheadrightarrow (\mu_1, \mu_0)$, $\nu \leq \kappa_0$, *and* $\kappa_0^\nu \geq \kappa_1$, *then* $\mu_0^{\min(\mu_0, \nu)} \geq \mu_1$.

Global Chang's Conjecture

For all infinite cardinals $\mu < \kappa$ with $\mathrm{cf}(\mu) \leq \mathrm{cf}(\kappa)$, $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$.

Approximations to GCC

Theorem (E.-Hayut)

It is consistent relative to a huge cardinal that $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$ *holds whenever* $\omega \leq \mu < \kappa$ *and* κ *is regular.*

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Theorem (E.-Hayut)

It is consistent relative to a huge cardinal that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ *while for all* $n < m < \omega$, $(\aleph_{m+1}, \aleph_m) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$.

It turns out that this was optimal; it is the longest initial segment of cardinals on which GCC can hold.

We say $(\kappa_1, \kappa_0) \twoheadrightarrow_\nu (\mu_1, \mu_0)$ holds when for all $\mathfrak{A}$ on κ_1, there is $\mathfrak{B} \prec \mathfrak{A}$ of size μ_1 with $|\mathfrak{B} \cap \kappa_0| = \mu_0$, and $\nu \subseteq \mathfrak{B}$. This is preserved under ν^+-c.c. forcing.

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Lemma

Suppose $(\kappa_1, \kappa_0) \twoheadrightarrow_\nu (\mu_1, \mu_0)$.

1. *If* $\kappa_0 = \mu_0^{+\nu}$, *then* $(\kappa_1, \kappa_0) \twoheadrightarrow_{\mu_0} (\mu_1, \mu_0)$.
2. *If* $\lambda \leq \mu_0$ *and there is* $\kappa \leq \kappa_0$ *such that* $\kappa_0 = \kappa^{+\nu}$ *and* $\kappa^\lambda \leq \kappa_0$, *then* $(\kappa_1, \kappa_0) \twoheadrightarrow_\lambda (\mu_1, \mu_0)$.

Lemma

Suppose $\mu^{<\nu} = \mu$, *and* $(\kappa^+, \kappa) \twoheadrightarrow (\mu^+, \mu)$. *Then* $(\kappa^+, \kappa) \twoheadrightarrow_\nu (\mu^+, \mu)$.

Scales

If κ is a singular cardinal, and $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ is an increasing sequence of regular carindals cofinal in κ, $\langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod_{i<\mathrm{cf}(\kappa)} \kappa_i$ is a *scale* for κ if it is increasing and dominating in the product (mod bounded). Shelah proved that singular κ always carry scales of length κ^+.

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A scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is *good at* α when there is a pointwise increasing sequence $\langle g_i : i < \mathrm{cf}(\alpha) \rangle$ such that this sequence and $\langle f_\beta : \beta < \alpha \rangle$ are cofinal in each other. A scale is *bad at* α when it is not good at α.

A scale is simply called *good* if it is good at every α such that $\mathrm{cf}(\alpha) > \mathrm{cf}(\kappa)$.

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Lemma (Folklore)

If κ is singular and $(\kappa^+, \kappa) \twoheadrightarrow_{\mathrm{cf}(\kappa)} (\mu^+, \mu)$ and $\mu \geq \mathrm{cf}(\kappa)$, then there is no good scale for κ. Moreover, every scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ for κ is bad at stationarily many α of cofinality μ^+.

Conflict at singulars

Lemma (E.-Hayut)

Suppose κ *is singular and* $(\kappa^{++}, \kappa^{+}) \twoheadrightarrow (\kappa^{+}, \kappa)$. *Then* κ *carries a good scale.*

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Suppose κ is singular and $(\kappa^{++}, \kappa^{+}) \twoheadrightarrow (\kappa^{+}, \kappa)$. Then κ carries a good scale.

We use a few known results. First due to Shelah: If $\mu < \kappa$ are regular, $S^{\kappa^+}_{\mu}$ is the union of κ sets each carrying a partial square.

Corollary

If κ is regular, then there is a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^{+}, \mathrm{cf}(\alpha) < \kappa \rangle$ forming a "partial weak square."

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Lemma (Foreman-Magidor)

For all κ, there is a structure $\mathfrak{A}$ on κ^{++} such that any $\mathfrak{B} \prec \mathfrak{A}$ witnessing $(\kappa^{++}, \kappa^{+}) \twoheadrightarrow_{\kappa} (\kappa^{+}, \kappa)$ has $\mathrm{cf}(\mathfrak{B} \cap \kappa^{+}) = \mathrm{cf}(\kappa)$.

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We use Chang's Conjecture to transfer the partial weak square on κ^{++} to one on κ^{+} that is defined at every ordinal of cofinality $> \mathrm{cf}(\kappa)$.

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How? If $\mathfrak{B} \prec (H_{\kappa^{+2}}, \in, \langle \mathcal{C}_\alpha : \alpha < \kappa^{++} \rangle)$ witnesses CC, then:

1. $\mathrm{ot}(\mathfrak{B} \cap \kappa^{++}) = \kappa^+$.
2. $|\mathcal{C}_\alpha \cap \mathfrak{B}| \leq \kappa$ for all $\alpha \in \mathfrak{B} \cap \kappa^{++}$.
3. $C \cap \mathfrak{B} = C$ for any $C \in \mathcal{C}_\alpha \in \mathfrak{B}$.
4. $\mathfrak{B} \cap \alpha$ is cofinal in α iff $\mathrm{cf}(\alpha) \neq \kappa^+$.

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This is enough to carry out the well-known construction of a good scale from weak square.

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Theorem (E.-Hayut)

Assume GCH. Suppose $\alpha < \beta$ are countable limit ordinals and κ is $\kappa^{+\beta+1}$-supercompact. Then there is a forcing extension in which $(\aleph_{\beta+1}, \aleph_\beta) \twoheadrightarrow (\aleph_{\alpha+1}, \aleph_\alpha)$.

CC between any two singulars below $\aleph_{\omega_1}$

The proof of the second consistency result breaks into cases depending on the "tail types" α and β. For ordinals $\alpha \geq \beta$, let $\alpha - \beta$ be the unique γ such that $\alpha = \beta + \gamma$. For an ordinal α, let $\tau(\alpha)$ (the tail of α) be $\min_{\beta<\alpha}(\alpha - \beta)$. Let $\iota(\alpha)$ be the least β such that $\alpha = \beta + \tau(\alpha)$. An ordinal α is indecomposable iff $\alpha = \tau(\alpha)$, and all tails are indecomposable.

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Lemma

Let $\eta < \kappa$ be such that $\kappa^{+\eta}$ is a strong limit cardinal and κ is $\kappa^{+\eta+1}$-supercompact, as witnessed by an embedding $j : V \to M$. If $\mathcal{U}$ is the ultrafilter on κ derived from j, then there is $A \in \mathcal{U}$ such that for every $\alpha < \beta$ in $A \cup \{\kappa\}$ and every iteration $\mathbb{P} * \dot{\mathbb{Q}}$ of size $< \beta^{+\eta}$, such that $\mathbb{P}$ is $\alpha^{+\eta+1}$-Knaster and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $(\alpha^{+\eta+1}, \alpha^{+\eta+1})$-distributive,

$$\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} (\beta^{+\eta+1}, \beta^{+\eta}) \twoheadrightarrow_{\alpha^{+\eta}} (\alpha^{+\eta+1}, \alpha^{+\eta}).$$

CC between any two singulars below $\aleph_{\omega_1}$, Case 1

Case 1: $\tau(\alpha) = \tau(\beta) = \gamma$, or $\alpha = 0$. Let $A \subseteq \kappa$ be given by the lemma (with respect to γ). Let $\delta = \iota(\beta) - \alpha$. Let $\zeta < \eta$ be in A, and force with $\mathrm{Col}(\zeta^{+\gamma+\delta+2}, \eta)$.

By the lemma we have $(\eta^{+\gamma+1}, \eta^{+\gamma}) \twoheadrightarrow_{\zeta^{+\gamma}} (\zeta^{+\gamma+1}, \zeta^{+\gamma})$. Next, If $\alpha = 0$, force with $\mathrm{Col}(\omega, \zeta^{+\gamma})$, and if $\alpha > 0$, force with $\mathrm{Col}(\aleph_{\iota(\alpha)+1}, \zeta)$.

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For the other cases, we will use a variation on the Gitik-Sharon forcing. Suppose $\gamma < \delta$ are ordinals of countable cofinality, with $\tau(\delta) > \gamma$, and κ is $\kappa^{+\gamma}$-supercompact, The forcing we call $\mathbb{P}(\mu^{+\delta}, \kappa^{+\gamma})$ is (μ, μ)-distributive, turns κ into $\mu^{+\delta}$, collapses all cardinals in the interval $(\kappa, \kappa^{+\gamma}]$, and have the $\kappa^{+\gamma+1}$-c.c.

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Let $\langle \gamma_i : i < \omega \rangle$ and $\langle \delta_i : i < \omega \rangle$ be increasing cofinal sequences in γ and δ respectively, with $\gamma < \delta_0$. Since $\tau(\delta) > \gamma$, we may assume that for all i, $\delta_i + \gamma < \delta_{i+1}$. Let $\delta'_0 = \delta_0$ and for each $i > 0$, let $\delta'_{i+1} = \delta_{i+1} - \delta_i$.

For each $n < \omega$, let U_n be a normal measure on $\mathcal{P}_\kappa(\kappa^{+\gamma_n})$. For each n, let $j_n : V \to M_n \cong \mathrm{Ult}(V, U_n)$ be the ultrapower embedding. By the closure of the ultrapowers and GCH, we may choose an M_n-generic $G_n \subseteq \mathrm{Col}(\kappa^{\delta'_n+2}, j_n(\kappa))^{M_n}$. Conditions in the forcing are sequences

$$\langle f_0, x_1, f_1, \ldots, x_{n-1}, f_{n-1}, F_n, F_{n+1}, \ldots \rangle,$$

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3. $f_0 \in \mathrm{Col}(\mu, \kappa_1)$.
4. For $1 \le i < n-1$, $f_i \in \mathrm{Col}(\kappa_i^{+\delta'_i+2}, \kappa_{i+1})$.
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5. $f_{n-1} \in \mathrm{Col}(\kappa_i^{+\delta'_i+2}, \kappa)$.
6. For $i \geq n$, $\mathrm{dom}\, F_i \in U_i$.
7. For $i \geq n$ and $x \in \mathrm{dom}\, F_i$, $\kappa_x := x \cap \kappa$ is a cardinal greater than $|x_{i-1}| + \sup(\mathrm{ran}\, f_{i-1})$.
8. For $i \geq n$ and $x \in \mathrm{dom}\, F_i$, $F_i(x) \in \mathrm{Col}(\kappa_x^{+\delta'_i+2}, \kappa)$.
9. For $i \geq n$, $[F_i]_{U_i} \in G_i$.

CC between any two singulars below $\aleph_{\omega_1}$, Case 2

Case 2: $\tau(\alpha) > \tau(\beta) = \gamma$. Again, we have $\iota(\beta) \geq \alpha$, so let $\delta = \iota(\beta) - \alpha$. Let $A \subseteq \kappa$ be given by the lemma (with respect to γ). Find $\nu < \mu$ in A such that ν is $\nu^{+\gamma}$-supercompact. Let $G \subseteq \mathrm{Col}(\nu^{+\gamma+\delta+2}, \mu)$ be generic over V. In $V[G]$, $(\mu^{+\gamma+1}, \mu^{+\gamma}) \twoheadrightarrow_{\nu^{+\gamma}} (\nu^{+\gamma+1}, \nu^{+\gamma})$ holds, and ν is still $\nu^{+\gamma}$-supercompact.

Then let $H \subseteq \mathbb{P}(\omega^{+\alpha}, \nu^{+\gamma})$ be generic over $V[G]$. In $V[G][H]$, CC is preserved, $\nu = \aleph_\alpha$ and $\mu^{+\gamma} = \aleph_{\alpha+\delta+\gamma} = \aleph_\beta$.

CC between any two singulars below $\aleph_{\omega_1}$, Case 3

Case 3: $0 < \tau(\alpha) = \gamma < \tau(\beta)$. Let $\delta = \beta - \iota(\alpha)$. Let $A \subseteq \kappa$ be given by the lemma. Force with $\mathbb{P}((\aleph_{\iota(\alpha)+1})^{+\delta}, \kappa^{+\gamma})$. Let p_0 be a condition of length 1 deciding some $\lambda \in A$ to be the first Prikry point. Let $p_1 \leq^* p_0$ decide the statement $\sigma :=$ "$(\kappa^+, \kappa) \twoheadrightarrow (\lambda^{+\gamma+1}, \lambda^{+\gamma})$." We claim that $p_1 \Vdash \sigma$. It is forced that $\lambda^{+\gamma} = \aleph_\alpha$ and $\kappa = \aleph_{\iota(\alpha)+\delta} = \aleph_\beta$.

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Let $\langle U_n : n < \omega \rangle$ and $\langle G_n : n < \omega \rangle$ be the sequences of normal ultrafilters and generic filters over ultrapowers used in the construction of $\mathbb{P} = \mathbb{P}((\aleph_{\iota(\alpha)+1})^{+\delta}, \kappa^{+\gamma})$. Let us define an iteration of ultrapowers.

Let $N_0 = V$. Given a commuting system of elementary embeddings $j_{m,m'} : N_m \to N_{m'}$ for $m \leq m' \leq n$, let $j_{n,n+1} : N_n \to \mathrm{Ult}(N_n, j_{0,n}(U_{n+1})) = N_{n+1}$ be the ultrapower embedding, and let $j_{m,n+1} = j_{n,n+1} \circ j_{m,n}$ for $m < n$. For each $n < \omega$, let $j_{n,\omega} : N_n \to N_\omega$ be the direct limit embedding. N_ω is well-founded.

CC between any two singulars below $\aleph_{\omega_1}$, Case 3

Let $\text{stem}(p_1) = \langle f_0 x_1, f_1 \rangle$, and let $C_0 \times C_1 \subseteq \text{Col}(\aleph_{\alpha+1}, \lambda) \times \text{Col}(\lambda^{+\delta_0+2}, \kappa)$ be generic over V containing (f_0, f_1). Let $y_1 = j_{0,\omega}(x_1)$. For $n > 1$, let $x_n = j_{n-1,n}[j_{0,n-1}(\kappa^{+\gamma_n})]$, and let $y_n = j_{n,\omega}(x_n)$. Let $C_n = j_{0,n-1}(G_n)$.

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Claim 1: $\langle C_0, y_1, C_1, y_2, C_2, \ldots \rangle$ generates a generic for $j_{0,\omega}(\mathbb{P})$ over N_ω.

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Claim 2: Let G be the generated filter for $j_{0,\omega}(\mathbb{P})$. Then $N_\omega[G]$ is closed under κ-sequences from $V[C_0][C_1]$.

CC between any two singulars below $\aleph_{\omega_1}$, Case 3

Let $\text{stem}(p_1) = \langle f_0 x_1, f_1 \rangle$, and let $C_0 \times C_1 \subseteq \text{Col}(\aleph_{\alpha+1}, \lambda) \times \text{Col}(\lambda^{+\delta_0+2}, \kappa)$ be generic over V containing (f_0, f_1). Let $y_1 = j_{0,\omega}(x_1)$. For $n > 1$, let $x_n = j_{n-1,n}[j_{0,n-1}(\kappa^{+\gamma_n})]$, and let $y_n = j_{n,\omega}(x_n)$. Let $C_n = j_{0,n-1}(G_n)$.

Claim 1: $\langle C_0, y_1, C_1, y_2, C_2, \ldots \rangle$ generates a generic for $j_{0,\omega}(\mathbb{P})$ over N_ω.

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By GCH and some counting arguments, $j_{0,\omega}(\kappa) = \kappa^{+\gamma}$ and $j_{0,\omega}(\kappa^{+\gamma+1}) = \kappa^{+\gamma+1}$.

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By elementarity, p_1 forces $(\kappa^+, \kappa) \twoheadrightarrow (\lambda^{+\gamma+1}, \lambda^{+\gamma})$.

Singular GCC below $\aleph_{\omega^\omega}$

A cardinal δ is called *Woodin for supercompactness* when for every $A \subseteq \delta$ there is $\kappa < \delta$ such that for all $\lambda \in (\kappa, \delta)$, there is a normal κ-complete ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that $j_U(A) \cap \lambda = A \cap \lambda$.

Like Woodin cardinals, Woodin for supercompactness cardinals need not be even weakly compact, but they have higher consistency strength than supercompact cardinals. Every almost-huge cardinal is Woodin for supercompactness.

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Lemma

Suppose GCH and δ is $\delta^{+\omega+1}$-supercompact and Woodin for supercompactness. Then there is a model in which GCH holds, there is a supercompact cardinal κ, and there is some some ordinal $\alpha_0 < \kappa$ such that for all $\beta > \alpha \geq \alpha_0$, $(\beta^{+\omega+1}, \beta^{+\omega}) \twoheadrightarrow (\alpha^{+\omega+1}, \alpha^{+\omega})$. Furthermore, such instances of Chang's Conjecture are preserved by forcing over this model with any $(\alpha^{+\omega+1}, \alpha^{+\omega+1})$-distributive forcing of size $< \beta^{+\omega}$.

Singular GCC below $\aleph_{\omega^\omega}$

Starting from a model as above, we introduce a Radinized version of Gitik-Sharon forcing, which adds a club of ordertype ω^ω of former large cardinals, using a $(+\omega^2)$-supercompactness measure. We go as far as we can with "converting ordinal addition into ordinal multiplication."

We define some classes of forcings inductively. GS_1 is the collection of forcings of the form $\mathbb{P}(\mu^{+\omega^2}, \kappa^{+\omega})$.

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In the general case we work with sequences of ultrafilters paired with collapse filters $\langle U_\alpha, K_\alpha : \alpha < \omega \cdot n \rangle$ such that:

1. There is a $\kappa > \omega$ such that $\text{crit}(\langle U_\alpha, K_\alpha : \alpha < \omega \cdot n \rangle) = \kappa$.

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3. For $1 \leq m \leq n$, $\omega \cdot (m-1) \leq \alpha < \omega \cdot m$, if $j_\alpha : V \to M_\alpha$ is the ultrapower embedding from U_α, then K_α is $\mathrm{Col}(\kappa^{+\omega \cdot m+2}, j_\alpha(\kappa))^{M_\alpha}$-generic over M_α.

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Suppose $n > 1$, we have defined GS_m for $m < n$, and we have functions $\phi_m : H_\theta \to \{\emptyset\} \cup \mathrm{GS}_m$, where $\phi_m(\mu, d) \neq \emptyset$ only if μ is regular and d is an appropriate sequence of filters of length $\omega \cdot m$.

Conditions take the form:

$p = \langle f_0, e_1, (x_1, d_1, a_1), f_1, \ldots, e_\ell, (x_\ell, d_\ell, a_\ell), f_\ell, \vec{F} \rangle$.

1. For $i \leq \ell$:
   1. $|x_i| < \kappa$, $x_i \prec H_{\kappa^{+\omega\cdot(n-1)+i}}$, $\text{crit}(x_i) = x_i \cap \kappa$, the transtive collapse of x_i is $H_{(x_i \cap \kappa)^{+\omega\cdot(n-1)+i}}$, and $\langle U_\alpha, K_\alpha : \alpha < \omega \cdot (n-1) \rangle \in x_i$.
   2. d_i is a sequence $\langle u_\alpha, k_\alpha : \alpha < \omega \cdot (n-1) \rangle$ such that $\phi_{n-1}(\text{crit}(x_{i-1})^{+\omega\cdot n+2}, d_i) \in \text{GS}_{n-1}$, and $\text{crit}(d_i) = \text{crit}(x_i)$.
   3. If $\pi : x_i \to H$ is the transitive collapse map, then $\pi(\langle U_\alpha, K_\alpha : \alpha < \omega \cdot (n-1) \rangle) = d_i$.
   4. a_i is a sequence of functions $\langle b_\alpha : \alpha < \omega \cdot (n-1) \rangle$ such that $\text{dom}(b_\alpha) \in u_\alpha$ and $[b_\alpha]_{u_\alpha} \in k_\alpha$.
2. For $i \leq \ell$, $\langle f_{i-1} \rangle^\frown e_i^\frown a_i \in \phi_{n-1}(\text{crit}(x_{i-1})^{+\omega\cdot n+2}, d_i)$.
3. For $i < \ell$, $\{x_i, f_i\} \in x_{i+1}$, and $|x_i| < \min(e_{i+1})$.
4. $f_\ell \in \text{Col}((x_\ell \cap \kappa)^{+\omega\cdot n+2}, \kappa)$.
5. $\vec{F}$ is a sequence of functions $\langle F_\alpha : \alpha < \omega \cdot n \rangle$ such that for each α, $\text{dom}\, F_\alpha \in U_\alpha$ and $[F_\alpha]_{U_\alpha} \in K_\alpha$.

Singular GCC below $\aleph_{\omega^\omega}$

Finally, we define GS_ω by diagonally weaving together the GS_n. This gets what we want. For the argument, we iterate ultrapowers ω^n many times, for each n.

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1. We need a model with a supercompact and lots of long intervals of cardinals on which SGCC holds. But the longest we can get these is ω^2 (containing ω many singulars).

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3. Or perhaps it is due to some mystical property of ω^ω. After all...

Question

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Thank you for your attention!