Killing ideals softly

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- $\mathrm{cov}([\omega]^\omega, \mathcal{I}) > \omega$.

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- $\mathbb{P}$ destroys $\mathrm{Fin} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall^\infty n \ \ ((A)_n \text{ is finite})\}$ iff $\mathbb{P}$ adds a dominating real.
- If $\mathbb{P}$ adds new reals, then it destroys $\mathrm{Conv} = \mathrm{id}\{C \subseteq \mathbb{Q} : C \text{ is convergent in } \mathbb{R}\}$.

The Hrušák-Zapletal characterization

Let I be a σ-ideal on ${}^{\omega}2$ (or on ${}^{\omega}\omega$), then the **trace** of I, an ideal on ${}^{<\omega}2$ (on ${}^{<\omega}\omega$ resp.), is defined as follows:

$$\mathrm{tr}(I) = \Big\{A \subseteq {}^{<\omega}2 : \underbrace{\{x \in {}^{\omega}2 : \exists^{\infty} n\ x \restriction n \in A\}}_{=[A]_{\delta},\ \text{the } G_{\delta}\text{-closure of } A} \in I\Big\}.$$

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Notice that $\mathbb{P}_I := \mathrm{Borel}({}^{\omega}2) \setminus I$ destroys $\mathrm{tr}(I)$. For example, $\mathrm{tr}(\mathcal{M}) \simeq \mathrm{Nwd}$; $\mathrm{tr}(\mathcal{N})$ is a tall Borel P-ideal, $\mathcal{I}_{1/n} \subseteq \mathrm{tr}(\mathcal{N}) \subseteq \mathcal{Z}$; and $\mathrm{tr}(\mathcal{K}_\sigma)$ is a coanalytic ideal.

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Theorem

Assume that $\mathbb{P}_I$ is proper and I satisfies the continuous reading of names. Then $\mathbb{P}_I$ can destroy an ideal $\mathcal{S}$ iff $\mathcal{S} \leq_{\mathrm{K}} \mathrm{tr}(I) \restriction X$ for some $X \in \mathrm{tr}(I)^{+}$, that is, there are an $X \in \mathrm{tr}(I)^{+}$ and an $f : X \to \omega$ such that $f^{-1}[A] \in \mathrm{tr}(I)$ for every $A \in \mathcal{S}$.

How large can the destroying set be?

Mathias-Prikry and Laver-Prikry

$(s, F) \in \mathbb{M}(\mathcal{I}^*)$ if $s \in [\omega]^{<\omega}$ and $F \in \mathcal{I}^*$; $(s_0, F_0) \leq (s_1, F_1)$ if s_0 end-extends s_1 in F_1 (i.e. $s_0 \setminus s_1 \subseteq F_1$) and $F_0 \subseteq F_1$.

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$T \in \mathbb{L}(\mathcal{I}^*)$ if $T \subseteq {}^{<\omega}\omega$ is a tree such that $\{n : t^\frown(n) \in T\} \in \mathcal{I}^*$ for every $t \in T$ above $\mathrm{stem}(T)$; $T_0 \leq T_1$ if $T_0 \subseteq T_1$.

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Observation

The $\mathbb{M}(\mathcal{I}^*)$-generic set is $\mathcal{I}$-positive for $\mathcal{I} = \mathrm{Nwd}, \mathcal{I}_{1/n}, \mathcal{Z}$, and Conv, and it belongs to $\mathcal{I}$ for $\mathcal{I} = \mathrm{Fin} \otimes \mathrm{Fin}$.

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$+$-destroying ideals

Definition

- We say that $\mathbb{P}$ **can $+$-destroy** the Borel ideal $\mathcal{I}$, if $\mathbb{P}$ adds an $\dot{H} \in \mathcal{I}^+$ such that

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- If $\mathcal{I}$ can be $+$-destroyed then $\text{cov}(\mathcal{I}^+, \mathcal{I}) > \omega$.
- $\text{non}(\mathcal{I}^+, \mathcal{I}) \geq \text{non}([\omega]^\omega, \mathcal{I})$ and $\text{cov}(\mathcal{I}^+, \mathcal{I}) \leq \text{cov}([\omega]^\omega, \mathcal{I})$.

Examples

$\mathrm{Conv} = \{A \subseteq \mathbb{Q} : |A'| < \omega\}$

We know that non$([\omega]^\omega, \mathrm{Conv}) = \omega$ and cov$([\omega]^\omega, \mathrm{Conv}) = \mathfrak{c}$.

(1) non$(\mathrm{Conv}^+, \mathrm{Conv}) = \omega$ and cov$(\mathrm{Conv}^+, \mathrm{Conv}) = \mathfrak{c}$.

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Which Borel ideals are ($+$-)destroyed by adding any new reals?

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Is it true that destroying an F_σ ideal implies $+$-destroying it?

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(2a) If $\mathbb{P}$ adds Cohen reals then it destroys Nwd. If $\mathbb{P}$ $+$-destroys Nwd then it adds dominating and Cohen reals.

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(2b) If $\mathbb{P}$ adds a Cohen real and $\Vdash_{\mathbb{P}}$"$\dot{\mathbb{Q}}$ adds a dominating real", then $\mathbb{P} * \dot{\mathbb{Q}}$ $+$-destroys Nwd.

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(2c) If $\mathbb{P}$ has the Laver property then $\mathbb{P}$ cannot destroy Nwd and $\mathbb{P} * \mathbb{C}$ cannot $+$-destroy Nwd.

Covering properties

Reformulation

Let $\mathcal{S}$ be a Borel ideal and I a σ-ideal on a Polish space X such that $\mathbb{P}_I$ is proper. Then the following holds:

- $\mathbb{P}_I$ cannot destroy $\mathcal{S}$ iff whenever $(B_n)_{n\in\omega}$ is an infinite-fold cover of an I-positive set by Borel sets, that is,

  $$\{x \in X : \{n \in \omega : x \in B_n\} \text{ is infinite}\} \in I^+,$$

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then there is an $S \in \mathcal{S}$ such that $(B_n)_{n\in S}$ is an infinite-fold cover of an I-positive set.
- $\mathbb{P}_I$ cannot $+$destroy $\mathcal{S}$ iff whenever $(B_n)_{n\in\omega}$ is an $\mathcal{S}^+$-fold cover of an I-positive set by Borel sets, that is,
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Forcing with $\mathbb{M}(\mathcal{I}^*)$ and $\mathbb{L}(\mathcal{I}^*)$

Theorem

Let $\mathcal{I}$ be a tall Borel ideal. Then the following are equivalent:
(a) The $\mathbb{M}(\mathcal{I}^*)$-generic $+$-destroys $\mathcal{I}$. (b) $\mathbb{M}(\mathcal{I}^*)$ $+$-destroys $\mathcal{I}$.
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Proofs: Apply Laflamme's characterization of winning strategies in the games $G(\mathcal{I}^*, [\omega]^{<\omega}, \mathcal{I}^+)$ and $G(\mathcal{I}^*, \omega, \mathcal{I}^+)$.

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There is a sequence $(\dot{F}_m)_{m\in\omega}$ s.t. $T_0 \Vdash$"$\dot{F}_m \subseteq \dot{X} \cap [2^n, 2^{n+1})$ and $|\dot{F}_m|/2^n > \varepsilon$ for some n, and $\max(\dot{F}_m) < \min(\dot{F}_{m+1})$".

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An $s \in \mathrm{Split}(T_0)$ **favors** $\dot{F}_m = E$ if

$$\forall\, T \le T_0\ (\mathrm{stem}(T) = s \longrightarrow \exists\, T' \le T\ T' \Vdash \dot{F}_m = E).$$

Define ϱ_m ($m \in \omega$) on $\mathrm{Split}(T_0)$: $\varrho_m(s) = 0$ if there is an E^s_m such that s favors $\dot{F}_m = E^s_m$; and $\varrho_m(s) = \alpha > 0$ if $\varrho_m(s) \not< \alpha$ and $\{n : \varrho_m(s^\frown(n)) < \alpha\} \in \mathcal{Z}^+$. Then $\mathrm{dom}(\varrho_m) = \mathrm{Split}(T_0)$ and (w.l.o.g.) $\varrho_m(s) > 0$ for every $m \ge |s|$.

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If $\varrho_m(s) = 1$ then define

$f_{m,s} : Y_{m,s} = \{n : \varrho_m(s^\frown(n)) = 0\} \to \bigcup_{n\in\omega} \{E \subseteq P_n : |E|/2^n > \varepsilon\}$,

$f_{m,s}(n) = E^{s^\frown n}_m$, i.e. $s^\frown(n)$ favors $\dot{F}_m = f_{m,s}(n)$ (and $Y_{m,s} \in \mathcal{Z}^+$).

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There is an $A \in \mathcal{Z}$ s.t. $Y'_{m,s} = \{n \in Y_{m,s} : A \cap f_{m,s}(n) \neq \emptyset\} \in \mathcal{Z}^+$ whenever $\varrho_m(s) = 1$. We claim that $T_0 \Vdash |\dot{X} \cap A| = \omega$.

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Let $T \le T_0$, $\mathrm{stem}(T) = t$, and $M \in \omega$. Fix an $m \ge M, |t|$, then $\varrho_m(t) > 0$ and hence there is a $s \in T \cap t^\uparrow$ of m-rank 1, and so an $n \in Y'_{m,s}$ such that $s^\frown(n) \in T$.

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There is an $A \in \mathcal{Z}$ s.t. $Y'_{m,s} = \{n \in Y_{m,s} : A \cap f_{m,s}(n) \neq \emptyset\} \in \mathcal{Z}^+$ whenever $\varrho_m(s) = 1$. We claim that $T_0 \Vdash |\dot{X} \cap A| = \omega$.
Let $T \le T_0$, $\mathrm{stem}(T) = t$, and $M \in \omega$. Fix an $m \ge M, |t|$, then $\varrho_m(t) > 0$ and hence there is a $s \in T \cap t^\uparrow$ of m-rank 1, and so an $n \in Y'_{m,s}$ such that $s^\frown(n) \in T$. As $s^\frown(n)$ favors $\dot{F}_m = f_{m,s}(n)$, there is a $T' \le T{\restriction}(s^\frown(n))$ which forces $\dot{F}_m = f_{m,s}(n)$, and we know that $A \cap f_{m,s}(n) \neq \emptyset$ and of course $f_{m,s}(n) \subseteq \omega \setminus M$.

Thank you for your attention!