

The Spectrum of Independence

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Independence Number

A family $\mathcal{A} = \{A_i\}_{i \in I} \subseteq [\omega]^\omega$ is said to be independent if whenever S, T are disjoint finite subsets of I , the set $\bigcap_{s \in S} A_s \cap (\omega \setminus \bigcup_{t \in T} A_t)$ is infinite. It is a m.i.f. if it is maximal under inclusion.

$$i = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a m.i.f.}\}$$

Boolean combinations

Let \mathcal{A} be a independent and let $\text{FF}(\mathcal{A})$ be the set of all finite partial functions from \mathcal{A} to 2 . For $h \in \text{FF}(\mathcal{A})$ define

$$\mathcal{A}^h = \bigcap \{A : A \in h^{-1}(0)\} \cap (\omega \setminus \bigcup \{A : A \in h^{-1}(1)\}).$$

$$\text{BC}(\mathcal{A}) := \{\mathcal{A}^h : h \in \text{FF}(\mathcal{A})\}$$

Countable independent families are not maximal

Let \mathcal{A} be a countable independent family and let $\{h_n\}_{n \in \omega}$ be an enumeration of $\text{FF}(\mathcal{A})$ so that each element appears cofinally often. Inductively define $\{a_{2n}, a_{2n+1}\}_{n \in \omega}$ so that

$$a_{2n}, a_{2n+1} \text{ belong to } \mathcal{A}^{h_n} \setminus \{a_{2k}, a_{2k+1}\}_{k < n}.$$

Then $A = \{A_{2n}\}_{n \in \omega}$ is independent over \mathcal{A} .

Fichtenholz-Kantorovich

Let $C = [\mathbb{Q}]^{<\omega}$ and for $r \in \mathbb{R}$ let

$$A_r = \{a \in C : a \cap (-\infty, r] \text{ is even}\}.$$

Then whenever S, T are finite disjoint sets of reals, the set

$$\bigcap_{r \in S} A_r \cap (C \setminus \bigcup_{r \in T} A_r)$$

is infinite. Thus, there is always a m.i.f. of size c .

Lower bounds

- If \mathcal{A} is m.i.f. then $\text{BC}(\mathcal{A})$ is an un-reaped family. Thus $\tau \leq i$.
- Also $\delta \leq i$.

Almost disjointness number

An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is said to be maximal almost disjoint if its elements are pairwise almost disjoint and is maximal under inclusion.

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a mad family}\}$$

The ultrafilter number

The ultrafilter number, denoted \mathfrak{u} , is defined as the minimal size of an ultrafilter base.

Together with \mathfrak{u} , \mathfrak{a} and \mathfrak{i} are the largest classical combinatorial cardinal characteristics of \mathbb{R} with only known upper bound \mathfrak{c} .

α VS. \mathfrak{u}

In the Cohen model $\alpha < \mathfrak{u}$, while assuming the existence of a measureable one can show the consistency of $\mathfrak{u} < \alpha$.

 i VS. \mathfrak{u}

In the Miller model $\mathfrak{u} < i$, while Shelah devised a special ${}^\omega\omega$ -bounding poset the countable support iteration of which produces a model of $i = \aleph_1 < \mathfrak{u} = \aleph_2$.

\aleph VS i

In the Cohen model $\aleph < i = \mathfrak{c}$.

The consistency of $i < \aleph$ is open.

Spectrum

Let $\text{Sp}(i) = \{|\mathcal{A}| : \mathcal{A} \text{ is independent}\}$. Analogously, one can define $\text{Sp}(a)$, $\text{Sp}(a_g)$, $\text{Sp}(a_e)$. Even though $\text{Sp}(a)$ has been extensively studied, there is very little known about $\text{Sp}(i)$. Thus one can reformulate the questions and ask more generally:

- Is it consistent that $\min \text{Sp}(i) < \min \text{Sp}(a)$?
- Is it consistent that $\text{Sp}(a)$ is a proper subset of $\text{Sp}(i)$?

Isomorphism of names for the Cohen poset

Lemma

Assume CH. Let \mathbb{C}_λ be the poset for adjoining λ Cohen reals, where λ is regular uncountable. Then in $V^{\mathbb{C}_\lambda}$ there are no m.a.d. family of size κ , where $\aleph_1 < \kappa < \lambda$.

Proof:

Assume $\{\dot{A}^\alpha\}_{\alpha < \kappa}$ are names for subsets of ω . Identify \dot{A}^α with a family $\{(p_{n,i}^\alpha, k_{n,i}^\alpha)\}_{n,i \in \omega}$ where for each α and n , $\{p_{n,i}^\alpha\}_{i \in \omega}$ is a maximal antichain, $\{k_{n,i}^\alpha\} \in 2$ and

- $p_{n,i}^\alpha \Vdash \check{n} \in \dot{A}^\alpha$ iff $k_{n,i}^\alpha = 1$, and
- $p_{n,i}^\alpha \Vdash \check{n} \notin \dot{A}^\alpha$ iff $k_{n,i}^\alpha = 0$.

Proof (cnt'd):

Let $B^\alpha = \bigcup \{\text{dom}(p_{n,i}^\alpha) : i, n \in \omega\}$. Then $|\bigcup_\alpha B^\alpha| \leq \kappa < \lambda$. By CH and the Δ -system Lemma, we may assume that

- $\{B^\alpha\}_{\alpha < \omega_2}$ forms a Δ -system with root R , and
- $\varphi_{\alpha,\beta} : B^\alpha \rightarrow B^\beta$ is a bijection s.t. $\varphi_{\alpha,\beta} \upharpoonright R = \text{id}$

Then $\varphi_{\alpha,\beta}$ induces an isomorphism $\psi_{\alpha,\beta} : \mathbb{C}_{B^\alpha} \rightarrow \mathbb{C}_{B^\beta}$. Since there are $2^{\aleph_0} = \aleph_1$ many isomorphism types of names, we may assume that ψ sends \dot{A}^α to \dot{A}^β , i.e. $k_{n,i} = k_{n,i}^\alpha = k_{n,i}^\beta$ and $\psi(p_{n,i}^\alpha) = \psi_{\alpha,\beta}(p_{n,i}^\alpha) = p_{n,i}^\beta$.

Proof (cnt'd):

Define a new name \dot{A}^κ by choosing a set B^κ such that

- $B^\kappa \cap \bigcup_{\alpha < \kappa} B^\alpha = R$
- $\forall \alpha < \omega_2, \exists \varphi_{\alpha, \kappa} : B^\alpha \rightarrow B^\kappa$ bijection such that $\varphi_{\alpha, \beta} \upharpoonright R = \text{id}$.

Let $\psi_{\alpha, \kappa} : \mathbb{C}_{B^\alpha} \rightarrow \mathbb{C}_{B^\kappa}$ be the induced isomorphism. For each n, i define $p_{n,i}^\kappa = \psi_{\alpha, \kappa}(p_{n,i}^\alpha)$, $k_{n,i}^\kappa = k_{n,i}$ and take \dot{A}^κ to be the name $\{(p_{n,i}^\kappa, k_{n,i}^\kappa)\}_{n,i}$.

Proof (cnt'd):

Let $\beta < \kappa$ be arbitrary. There is $\alpha < \omega_2$ such that $B^\alpha \cap B^\beta \subseteq R$. Then $B^\kappa \cap B^\beta = B^\alpha \cap B^\beta$. Thus the canonical mapping extending $\varphi_{\alpha, \kappa}$ and sending $B^\alpha \cup B^\beta$ to $B^\kappa \cup B^\beta$ is a bijection, inducing an isomorphism between $\mathbb{C}_{B^\alpha \cup B^\beta}$ and $\mathbb{C}_{B^\kappa \cup B^\beta}$. Therefore $\Vdash_{\mathbb{C}_\lambda}$ " $\dot{A}^\kappa \cap \dot{A}^\beta$ is finite" and so $\{\dot{A}^\alpha\}_{\alpha < \kappa}$ is not maximal. \square

Remark

Thus, $V^{\mathbb{C}_\lambda} \models \text{Sp}(a) = \{\aleph_1, c\} \wedge \text{Sp}(i) = \{c\}$.

Standard template extension

Since $\delta \leq i$, if we are to have $i < \alpha$, then in particular $\delta < \alpha$. However, in the standard template model $\aleph_2 \leq \delta < \alpha = i = \mathfrak{c}$ and so in particular

$$\text{Sp}(\alpha) = \text{Sp}(i) = \{\mathfrak{c}\}.$$

Adjoining witnesses to independence

\mathcal{A} -diagonalization filter

Let \mathcal{A} be an independent family. Then the Frechét filter \mathcal{F}_0 has the following two properties:

- $\forall F \in \mathcal{F}_0 \forall B \in \text{BC}(\mathcal{A}), F \cap B$ is infinite, and
- $\mathcal{F}_0 \cap \text{BC}(\mathcal{A}) = \emptyset$.

Definition

A filter \mathcal{U} is said to be an \mathcal{A} -diagonalization filter, if \mathcal{U} extends \mathcal{F}_0 and \mathcal{U} is maximal with respect to the above two properties.

Lemma (F., Shelah)

If \mathcal{U} is a \mathcal{A} -diagonalization filter and G is $\mathbb{M}(\mathcal{U})$ -generic and $x_G = \bigcup \{s : \exists F(s, F) \in G\}$, then:

- 1 $\mathcal{A} \cup \{x_G\}$ is independent
- 2 If $y \in ([\omega]^\omega \setminus \mathcal{A}) \cap V$ is such that $\mathcal{A} \cup \{y\}$ is independent, then $\mathcal{A} \cup \{x_G, y\}$ is not independent.

Proof (1):

For $h \in \text{FF}(\mathcal{A})$ and $n \in \omega$, the sets

- $D_{h,n} := \{(s, F) \in \mathbb{M}(\mathcal{U}) : |s \cap \mathcal{A}^h| > n\}$, and
- $E_{h,n} := \{(s, F) : |(\min F \setminus \max s) \cap \mathcal{A}^h| > n\}$

are dense, and so $\mathcal{A}^h \cap x_G$, and $\mathcal{A}^h \setminus x_G$ are infinite.

Proof (2):

Fix y such that $\mathcal{A} \cup \{y\}$ is independent.

- ① If $y \in \mathcal{U}$, then $x_G \subseteq^* y$ and so $x_G \setminus y$ is finite.
- ② If $y \notin \mathcal{U}$, then
 - either there is $F \in \mathcal{U}$ such that $F \cap y$ is finite, and so $x_G \cap y$ is finite,
 - or there are $F \in \mathcal{U}$, $h \in \text{FF}(\mathcal{A})$ s.t. $F \cap y \subseteq \mathcal{A}^h$. Let $C \in \text{dom}(h)$.
 Wlg $h(C) = 0$. Then $F \cap y \subseteq \mathcal{A}^h \subseteq C$, and so $x_G \cap y \cap (\omega \setminus C)$ is finite.



Definition

We say that y diagonalizes \mathcal{A} over V_0 (in V_1) iff

- 1 V_1 extends V_0 , (\mathcal{A} is independent) V_0
- 2 $y \in ([\omega]^{\aleph_0})^{V_1} \setminus \mathcal{A}$ such that $V_0 \models \mathcal{A} \cup \{x\}$ is independent, then $V_1 \models \mathcal{A} \cup \{x, y\}$ is not independent.

Corollary

Let \mathcal{A} be an independent family, \mathcal{U} an \mathcal{A} -diagonalization filter and let G be a $\mathbb{M}(\mathcal{U})$ -generic filter, then $\sigma_G = \bigcup \{s : \exists A(s, A) \in G\}$ diagonalizes \mathcal{A} over the ground model.

Analogous diagonalization properties are known for

- m.a.d. families (Solovay's Lemma) and
- m.c.g. (Yi Zhang),

and are used to adjoin a m.a.d. family, or resp. m.c.g. of desired cardinality along a finite support iteration.

A strengthening of the notion of diagonalization, referred to as strong diagonalization, has been developed in the context of matrix iterations (F., Brendle), to preserve the maximality of desired m.a.d. families in iterations along coherent systems.

Sp(i) can be large

Theorem (F., Shelah)

Assume GCH. Let $\kappa_1 < \dots < \kappa_n$ be regular uncountable cardinals. Then it is consistent that $\{\kappa_i\}_{i=1}^n \subseteq \text{Sp}(i)$.

Proof:

Let $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and for each $j = 1, \dots, n$ let $I_j \subseteq \gamma^*$ be such that:

- I_j is unbounded in γ^* , $|I_j| = \kappa_j$ and
- $\{I_j\}_{j=1}^n$ are pairwise disjoint.

Along I_j inductively construct a m.i.f. of cardinality κ_j . Define a fsit of length γ^* as follows. Fix $\alpha < \gamma$ and suppose for each $k \in \{1, \dots, n\}$ a sequence of reals $\langle r_\gamma^k : \gamma \in I_k, \gamma < \alpha \rangle$ has been defined such that

- $\mathcal{I}_\alpha^k = \cup \{r_\gamma^k : \gamma \in I_k \cap \alpha\}$ is independent, and
- for each $\gamma \in I_k$, r_γ^k diagonalizes $\mathcal{I}_\gamma^k = \cup \{r_\delta^k : \delta \in I_k \cap \gamma\}$ over $V^{\mathbb{P}_\gamma}$.

Proceed as follows. If $\alpha \in I_j$ for some $j \in \{1, \dots, n\}$ then

- choose an \mathcal{I}_α^j -diagonalizing filter \mathcal{U}_α in $V^{\mathbb{P}_\alpha}$,
- take \dot{Q}_α to be a \mathbb{P}_α -name for $\mathbb{M}(\mathcal{U}_\alpha)$, and
- r_α^j to be the associated Mathias generic real.

If $\alpha \notin \bigcup_{k=1}^n I_k$ take \dot{Q}_α to be a \mathbb{P}_α -name for the Cohen poset.



Remark

The above argument generalizes. Whenever λ is the intended size of the continuum and we can partition λ into θ -many disjoint sets $\langle I_j : j \in \theta \rangle$ such that $|I_j| = \sigma_j$ and I_j is cofinal in λ , we can use an appropriate bookkeeping to adjoin m.i.f. of size σ_j for each $j \in \theta$. Then consistently

$$\{\sigma_j : j \in \theta\} \subseteq \text{Spec}(mif).$$

Sp(i) can be small

Theorem (F., Shelah)

Assume GCH. Let λ be a cardinal of uncountable cofinality. Let G be \mathbb{P} -generic filter, where \mathbb{P} is the countable support product of Sacks forcing of length λ . Then $V[G] \models \text{Sp}(i) = \{\aleph_1, \lambda\}$.

No intermediate cardinalities

Lemma

In the above extension there are no m.i.f. of size κ , for $\aleph_1 < \kappa < \lambda$.

No intermediate cardinalities

Proof:

Fix κ , $\aleph_1 < \kappa < \lambda$. Let $p_\star \in \mathbb{P}$ force that $\{\tau_\alpha\}_{\alpha \in \kappa}$ is a m.i.f. For $\alpha < \aleph_2$, let $p_\alpha \leq p_\star$, $\mathcal{U}_\alpha \in [\lambda]^{\aleph_0}$ such that the support of p_α , $\text{dom}(p_\alpha) = \mathcal{U}_\alpha$ and below p_α , τ_α can be read continuously. Find $S \in [\omega_2]^{\aleph_2}$ such that

- $\langle \mathcal{U}_\alpha : \alpha \in S \rangle$ is a Δ -system with root \mathcal{U}_\star ,
- the sequence $\langle \text{otp}(\mathcal{U}_\alpha) : \alpha \in S \rangle$ is constant, and
- for $\alpha \neq \beta$ from S , if $\pi_{\alpha,\beta} : \mathcal{U}_\beta \rightarrow \mathcal{U}_\alpha$ is order preserving and onto, then $\pi_{\alpha,\beta} \upharpoonright \mathcal{U}_\star = \text{Id}_{\mathcal{U}_\star}$, $\pi_{\alpha,\beta}$ maps $\tau_\beta(\leq p_\beta)$ onto $\tau_\alpha(\leq p_\alpha)$.

Proof (cnt'd):

Each τ_α depends only on \aleph_1 many $\{p_{\alpha,i}\}_{i < \omega_1} \subseteq \mathbb{P}$. Let $W_\alpha = \bigcup_i \text{dom}(p_{\alpha,i})$, $W = \bigcup_{\alpha < \kappa} W_\alpha$. Then $|W| < \lambda$ and we can find $\mathcal{U} \subseteq \lambda$ such that $\text{otp}(\mathcal{U}) = \text{otp}(\mathcal{U}_\alpha)$ for $\alpha \in S$, $\mathcal{U} \cap W = \mathcal{U}_*$.

For $\alpha \in S$ let $\pi_{\alpha,*} : \mathcal{U} \rightarrow \mathcal{U}_\alpha$ be order preserving and onto. Then the condition $p = \pi_{\alpha,*}^{-1}(p_\alpha)$ and the name $\tau = \pi^{-1}(\tau_\alpha \upharpoonright \leq p_\alpha)$ satisfy

$$p \leq p_\alpha \text{ and } p \Vdash (\{\tau\} \cup \{\tau_\alpha : \alpha \in \kappa\} \text{ is independent}),$$

which contradicts $p_* \Vdash (\{\tau_\alpha : \alpha < \kappa\} \text{ is maximal})$. \square

Theorem (Eisworth, Shelah)

Assume CH. Then there is a m.i.f. which is indestructible by the countable support product and the countable support iteration of Sacks forcing.

Remark

The result can be extracted from Shelah's $i < \aleph_1$ consistency proof.

Definition: Density Ideal

Let \mathcal{A} be independent. Then

$$\text{Id}(\mathcal{A}) = \{X \subseteq \omega : \forall h \in \text{FF}(\mathcal{A}) \exists h' \supseteq h (\mathcal{A}^{h'} \cap X \text{ is finite})\}$$

is an ideal on ω to which we refer as density ideal associated to \mathcal{A} .

- 1 For $X \subseteq \omega$, let $\mathcal{D}(X) := \{h \in \text{FF}(\mathcal{A}) : X \cap \mathcal{A}^h \text{ is finite}\}$. Then
- 2 $\text{Id}(\mathcal{A}) = \{X \subseteq \omega : \mathcal{D}(X) \text{ is dense in } \text{FF}(\mathcal{A})\}$.

Sacks indestructibility can be described in terms of properties of the above ideal.

Countable Approximations

Definition

Let \mathbb{P} be the poset of (\mathcal{A}, A) where \mathcal{A} is a countable independent family, $A \in [\omega]^\omega$ and for all $h \in \text{FF}(\mathcal{A})$ ($|\mathcal{A}^h \cap A| = \omega$). Define $(\mathcal{A}_2, A_2) \leq (\mathcal{A}_1, A_1)$ iff $\mathcal{A}_2 \supseteq \mathcal{A}_1$ and $A_2 \subseteq^* A_1$.

Theorem (F., Montoya, 2017)

Assume CH and $2^{\aleph_1} = \aleph_2$. Let G be \mathbb{P} -generic. Then

$$\mathcal{A}_G = \bigcup \{ \mathcal{A} : \exists A(\mathcal{A}, A) \in G \}$$

is m.i.f. which remains maximal after the countable support iteration, as well as countable support product of Sacks forcing.

Crucial Lemma (Shelah, 1992)

Assume CH . Then, there is a strictly decreasing $\langle (\mathcal{A}_\alpha, A_\alpha) : \alpha < \omega_1 \rangle$ in \mathbb{P} such that if $\mathcal{A} = \bigcup_{\alpha \in \omega_1} \mathcal{A}_\alpha$ then:

- 1 for every partition \mathcal{E} of ω and every $h \in \text{FF}(\mathcal{A})$, there is $h' \in \text{FF}(\mathcal{A})$ such that $h' \supseteq h$, and either $\exists E \in \mathcal{E} (\mathcal{A}^{h'} \subseteq E)$ or $\forall E \in \mathcal{E} (|E \cap \mathcal{A}^{h'}| \leq 1)$.
- 2 for every partition \mathcal{E} of ω into finite sets, there is $\alpha \in \omega_1$ such that $\forall E \in \mathcal{E} (|E \cap A_\alpha| \leq 1)$.
- 3 for each $\alpha \in \omega_1$ there is $A \subseteq A_\alpha$ such that $A \in \mathcal{A}_{\alpha+2}$;
- 4 for each $X \in \text{Id}(\mathcal{A})$ there are unboundedly many $\alpha \in \omega_1$ such that $X \subseteq \omega \setminus A_\alpha$.

Sacks indestructible m.i.f.

Theorem

Let $\{(\mathcal{A}_\alpha, \mathbf{A}_\alpha) : \alpha \in \omega_1\}$ be a strictly decreasing sequence in \mathbb{P} satisfying properties (1) – (4) of the Crucial Lemma. Then $\mathcal{A} = \bigcup_{\alpha \in \omega_1} \mathcal{A}_\alpha$ is a maximal independent family, which remains maximal after the countable support product of Sacks forcing, as well as after the countable support iteration of Sacks forcing.

Dense maximality

Definition:

An independent family \mathcal{A} is densely maximal if for every $X \in [\omega]^\omega \setminus \mathcal{A}$ and every $h \in \text{FF}(\mathcal{A})$ there is $h' \in \text{FF}(\mathcal{A})$ for which either $X \cap \mathcal{A}^{h'}$ or $\mathcal{A}^{h'} \setminus X$ is finite.

Characterization of Sacks Indestructibility

Theorem(F., Montoya, 2018)

- 1 If \mathcal{A} is densely maximal independent family, then the dual filter $\text{Id}^*(\mathcal{A})$ of the density ideal is the unique \mathcal{A} -diagonalization filter.
- 2 If \mathcal{A} is densely maximal and $\text{Id}^*(\mathcal{A})$ is Ramsey then \mathcal{A} is indestructible by the countable support products and countable support iterations of Sacks forcing.

Sp(i) is not necessarily convex

Let κ a measurable and let $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ be a κ -complete ultrafilter. Let \mathbb{P} be a p.o. Then $\mathbb{P}^\kappa/\mathcal{D}$ consists of all equivalence classes

$$[f] = \{g \in {}^\kappa\mathbb{P} : \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{D}\}$$

and is supplied with the p.o. relation $[f] \leq [g]$ iff

$$\{\alpha \in \kappa : f(\alpha) \leq_{\mathbb{P}} g(\alpha)\} \in \mathcal{D}.$$

We can identify each $p \in \mathbb{P}$ with $[p] = [f_p]$, where $f_p(\alpha) = p$ for each $\alpha \in \kappa$ and so we can assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathcal{D}$.

Lemma

- ① The poset \mathbb{P} is a complete suborder of \mathbb{P}^κ/D if and only if \mathbb{P} is κ -cc. Thus, if \mathbb{P} is ccc, then $\mathbb{P} \triangleleft \mathbb{P}^\kappa/D$.
- ② If \mathbb{P} has the countable chain condition, then so does \mathbb{P}^κ/D .

Lemma

Let \mathcal{A} be a \mathbb{P} -name for an independent family of cardinality $\geq \kappa$. Then $\Vdash_{\mathbb{P}^\kappa/D} \mathcal{A}$ is not maximal.

Theorem

Let $\kappa_1 < \kappa_2 < \dots < \kappa_n$ be measurable cardinals witnessed by κ_j -complete ultrafilters $\mathcal{D}_j \subseteq \mathcal{P}(\kappa_j)$. Then there is a ccc generic extension in which

$$\{\kappa_i\}_{i=1}^n \subseteq \text{Spec}(mif) \text{ and } (\kappa_i, \kappa_{i+1}) \cap \text{Spec}(mif) = \emptyset$$

for each $1 \leq i < n$.

Proof:

Let $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and for each $j = 1, \dots, n$ let $I_j \subseteq \gamma^*$ be such that:

- I_j consists of successor ordinals, $I_j \cap \text{Even}$ and $I_j \cap \text{Odd}$ are unbounded in γ^* , $|I_j| = \kappa_j$ and
- $\{I_j\}_{j=1}^n$ are pairwise disjoint.

Define a fsit of length γ^* as follows. Fix $\alpha < \gamma$ and suppose for each $k \in \{1, \dots, n\}$ a sequence of reals $\langle r_\gamma^k : \gamma \in I_k \cap \text{Even}, \gamma < \alpha \rangle$ has been defined such that

- $\mathcal{I}_\alpha^k = \cup \{r_\gamma^k : \gamma \in I_k \cap \text{Even} \cap \alpha\}$ is independent, and
- for each $\gamma \in I_k \cap \text{Even}$, r_γ^k diagonalizes \mathcal{I}_γ^k over $V^{\mathbb{P}_\gamma}$.

Proof (cnt'd):

Proceed as follows.

- 1 If $\alpha \in I_k \cap \text{Even}$ for some $k \in \{1, \dots, n\}$ then
 - choose an \mathcal{I}_α^k -diagonalizing filter \mathcal{U}_α in $V^{\mathbb{P}_\alpha}$,
 - take \dot{Q}_α to be a \mathbb{P}_α -name for $\text{MI}(\mathcal{U}_\alpha)$, and
 - r_α^k to be the associated Mathias generic real.
- 2 If $\alpha \in I_k \cap \text{Odd}$ for some $k \in \{1, \dots, k\}$, then $\alpha = \beta + 1$ and take \dot{Q}_α to be a \mathbb{P}_β -name for the quotient \mathbb{R}_α , where $\mathbb{P}^{k_k} / \mathcal{D}_k = \mathbb{P}_\beta * \mathbb{R}_\alpha$.
- 3 If $\alpha \notin \bigcup_{k=1}^n I_k$ take \dot{Q}_α to be a \mathbb{P}_α -name for the Cohen poset. \square

Spectrum of Independence

Work in Progress (F., Shelah)

The Spectrum of Independence can be quite arbitrary. The forcing is based on a template like construction which allows to adjoin m.i.f. of desired size and has at the same time enough local homogeneity to prevent undesired sizes to appear in $\text{Sp}(i)$.

Definability of m.a.d. and m.i.f.

- (Mathias) There is no analytic m.a.d. family.
- (Miller) There is no analytic m.i.f.

In L , using the wellorder of the reals, one can inductively define a Σ_2^1 -definable m.a.d. family and a Σ_2^1 -definable m.i.f. Furthermore:

- (Törnquist) If there is a Σ_2^1 m.a.d. family, then there is a Π_1^1 m.a.d.
- (Brendle, Khomskii) If there is a Σ_2^1 m.i.f., then there is a Π_1^1 m.i.f.

In L there is a co-analytic m.a.d. family, as well as a co-analytic m.i.f.

Models of $\mathfrak{c} > \aleph_1$

- There is a co-analytic Cohen indestructible m.a.d. family.
- (F., Schritterser, Törnquist) There is a co-analytic Cohen indestructible maximal cofinitary group.

The existence of a co-analytic mad family (resp. co-analytic mcg) of size \aleph_1 is consistent with $\mathfrak{c} > \aleph_1$.

Models of $\mathfrak{c} > \aleph_1$

Theorem (Brendle, Khomskii)

In the Cohen model, there are no projective m.i.f.

Definable Spectra in the Cohen Extension

In the Cohen extension: $\text{Sp}(a) = \text{Sp}(a_g) = \text{Sp}(a_e) = \{\aleph_1, \mathfrak{c}\}$.
Furthermore,

- 1 \aleph_1 has a Π_1^1 witness in each of the above three cases;
- 2 \mathfrak{c} has a Borel witness in the latter two cases;

Definable m.i.f. in models of $\mathfrak{c} > \aleph_1$

Theorem (F., 2018)

In L , there is a co-analytic Sacks indestructible m.i.f., i.e. indestructible by countable support products and iterations of Sacks forcing.

Thus in the Sacks model $\text{Sp}(i) = \{\aleph_1, \mathfrak{c}\}$ and \aleph_1 is witnessed by a Π_1^1 -set. Can we additionally provide nice definability properties to a witness of $\mathfrak{c} \in \text{Sp}(i)$?

Remark

The consistency proof of Shelah, that $i < u$, can be modified to provide in addition a Π_1^1 witness to i .

Further definability in the Sacks model

Theorem (F., Schritterser, 2018)

- 1 If there is a Σ_2^1 -definable m.e.d. family of functions, then there is a Π_1^1 -definable m.e.d. family of functions.
- 2 In L , there is a Σ_2^1 -definable m.e.d. family which is indestructible by the countable support products of Sacks forcing.

Consistently $\text{Sp}(a_e) = \{\aleph_1, c\}$ and both cardinals are witnessed by Π_1^1 -sets.

Definable witnesses of intermediate size

Theorem(F.)

Let $\aleph_0 < \kappa_0 < \kappa_1 < \dots < \kappa_{n-1}$ regular uncountable cardinals. Then there is a cardinals preserving generic extension in which $\{\kappa_i\}_{i \in n} \subseteq \text{Sp}(\alpha)$ and each κ_i is witnessed by a Π_2^1 set.

- (F., Friedman, Törnquist) Analogous results hold for $\text{Sp}(\alpha_g)$ and $\text{Sp}(\alpha_e)$. In both of these cases, \aleph_1 is witnessed by a Borel set.
- In all cases above, can we additionally require that \aleph_1 is witnessed by a Π_1^1 set?

Thank you!