Borel ideals

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We use the natural identification $\mathcal{P}(\mathbb{N}) \simeq 2^{\mathbb{N}}$ and view all operations and relations such as $\subseteq$, $\cap$, $[\_]^{<\omega}$, etc, as defined on $2^{\mathbb{N}}$.

Defintion

A family $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal on $\mathbb{N}$ if the following holds

- if $x, y \in \mathcal{I}$ then $x \cup y \in \mathcal{I}$,
- if $y \subseteq x$ and $x \in \mathcal{I}$ then $y \in \mathcal{I}$.

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is *tall* if for every infinite $x \in 2^{\mathbb{N}}$ there is an infinite $y \subseteq x$ such that $y \in \mathcal{I}$.

Given two ideals $\mathcal{I}, \mathcal{J}$ on $\mathbb{N}$ we say that $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}(I) \in \mathcal{J}$, for every $I \in \mathcal{I}$.

Question(Hrušák)

Is there a tall Borel ideal minimal among all tall Borel ideals in the Katětov order $\leq_K$?

Observation

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ containing $\mathbf{FIN}$. Then

- $\mathbf{FIN} \leq_K \mathcal{I}$,
- $\mathcal{I} \leq_K \mathbf{FIN}$ if and only if $\mathcal{I}$ is not tall.

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ is F_σ, Borel, $\mathbf{\Sigma}^1_1$, etc, if $\mathcal{I}$ is such as a subset of the Polish space $2^{\mathbb{N}}$. Let $A \subseteq 2^{\mathbb{N}}$ then define

$$\mathcal{I}_A = \{x \in \mathbb{N} : \exists y_0 \dots y_{n-1} \in A \; x \subseteq \bigcup_{i<n} y_i\}.$$

If A is Borel then $\mathcal{I}_A$ is $\mathbf{\Sigma}^1_1$ and if A is closed then $\mathcal{I}_A$ is F_σ.

Proposition

For every F_σ ideal $\mathcal{I}$ there is a closed set K such that $\mathcal{I}_K = \mathcal{I}$.

We may view the hyperspace $K(2^{\mathbb{N}})$ as the space of codes of F_σ ideals.

$$K \in K(2^{\mathbb{N}}) \leftrightarrow \mathcal{I}_K$$

Theorem(G.-Hrušák)

Let $\mathcal{I}$ be a $\mathbf{\Sigma}^1_1$ tall ideal on $\mathbb{N}$. Then there is a F_σ ideal $\mathcal{J}$ such that

$$\mathcal{I} \nleq_K \mathcal{J}.$$

Proof.

Let $\mathcal{T} \subseteq K(2^{\mathbb{N}})$ be a set of all closed subsets of $2^{\mathbb{N}}$ that code tall F_σ ideal.

- Use the work of Becker-Kahane-Louveau to show that the set $\mathcal{T}$ is $\mathbf{\Pi}^1_2-$complete.
- The set $\{K \in K(2^{\mathbb{N}}) : \mathcal{I} \leq_K \mathcal{I}_K\}$ is $\mathbf{\Sigma}^1_2$.

□

Let $\mathcal{I}$ be a $\mathbf{\Sigma}^1_1$ (or Borel) tall ideal. Is there a reasonable way how to find any Borel tall ideal $\mathcal{J}$ such that $\mathcal{I} \not\leq_K \mathcal{J}$?
This leads (us) to the following notion:

Definition

We say that an ideal $\mathcal{I}$ on $\mathbb{N}$ has a *selector* if there is a Borel map $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that

- $S(x) \subseteq x$ for every $x \in 2^{\mathbb{N}}$,
- $x \in \mathcal{I}$ for every $x \in 2^{\mathbb{N}}$,
- if $x \in 2^{\mathbb{N}}$ is infinite then $S(x)$ is infinite.

Remark

If $\mathcal{I}$ has selector then it is tall and the tallness is witnessed effectively.

Definition

Let $c : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a Borel function. We define $\mathcal{I}_c$ to be an ideal on $2^{<\mathbb{N}}$ generated by

- antichains,
- $\{x \upharpoonright m : m \in c(x)^{-1}(n)\}$ for every $x \in 2^{\mathbb{N}}$ and every $n \in \mathbb{N}$,
- $A \subseteq \{x \upharpoonright m : m \in \mathbb{N}\}$ where $x \in \mathbb{N}$ and for every $n \in \mathbb{N}$ is $|A \cap c(x)^{-1}(n)| = 1$.

Theorem(G.-Hrušák)

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ that has a selector. Then there is a Borel function $c : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{I} \not\leq_K \mathcal{I}_c$.

Remark

- $\mathcal{I}_c$ is always Borel,
- the function $c = H \circ S \circ \Theta$ where S is the selector and H, Θ are two Borel functions fixed in advance.

All the known examples of tall Borel ideals admit a selector: Random ideal, Solecki ideal, $\mathbf{FIN} \times \mathbf{FIN}$, ... Moreover this notion is closed upwards in $\leq_K$.

Question

Does every tall Borel ideal admit a selector?

Theorem(G.-Uzcátegui)

Let $\mathcal{S} \subseteq K(2^{\mathbb{N}})$ be the set of those closed subsets of $2^{\mathbb{N}}$ such that $\mathcal{I}_K$ has a selector. Then $\mathcal{S}$ is $\mathbf{\Sigma}^1_2$.

Corollary

There is a tall F_σ ideal $\mathcal{I}$ without a selector.

Proof of Corollary.

Note that we have $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{T}$ is $\mathbf{\Pi}^1_2$−complete. $\square$

Let $c : [\mathbb{N}]^2 \to 2$. Let $\mathcal{I}$ be generated by all monochromatic subsets of $\mathbb{N}$. Then $\mathcal{I}$ is F_σ and by the Ramsey's Theorem it is tall. Since the "standard" proof of Ramsey's Theorem is "enough Borel" we can use it to find a selector for $\mathcal{I}$. More generally we have.

Proposition

Let $n \in \mathbb{N}$. There is a Borel function $S : 2^{[\mathbb{N}]^n} \times [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ such that $S(\mathcal{F}, x)$ is $\mathcal{F}$-monochromatic and $S(\mathcal{F}, x) \subseteq x$.

Question

What other Ramsey statements have "Borel proof" (Borel way of choosing the monochromatic infinite subset)?

Let $s \in [\mathbb{N}]^{<\mathbb{N}}$ and $t \in [\mathbb{N}]^{\leq\mathbb{N}}$. We write $s \sqsubseteq t$ when there is $n \in \omega$ such that $s = t \cap \{0, 1, \cdots, n\}$ and we say that s is an initial segment of t.

Theorem (Galvin)

Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ and $x \in [\mathbb{N}]^{\mathbb{N}}$. Then there is an infinite $y \subseteq x$ such that one of the following holds

- *for all $z \in [y]^{\mathbb{N}}$ there is $s \in \mathcal{F}$ such that $s \sqsubseteq z$,*
- $[y]^{<\mathbb{N}} \cap \mathcal{F} = \emptyset$.

A special type of subsets of $[\mathbb{N}]^{<\mathbb{N}}$ is as follows. We say that $\mathcal{B} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ is a *front* if

- every two elements of $\mathcal{B}$ are $\sqsubseteq$-incomparable,
- every $x \in [\mathbb{N}]^{\mathbb{N}}$ has an initial segment in $\mathcal{B}$.

Theorem (Nash-Williams)

Let $\mathcal{B}$ be a front on $\mathbb{N}$ and $\mathcal{F} \subseteq \mathcal{B}$. Then for every infinite $x \in [\mathbb{N}]^{\mathbb{N}}$ there is an infinite $y \subseteq x$ such that one of the following holds

- $[y]^{<\mathbb{N}} \cap \mathcal{B} \subseteq \mathcal{F}$,
- $[y]^{<\mathbb{N}} \cap \mathcal{F} = \emptyset$.

Theorem (G.-Uzcátegui)

Let $\mathcal{B}$ be a front and $\mathcal{F} \subseteq \mathcal{B}$. There is a Borel map $S : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ such that $S(x) \subseteq x$ and $S(x)$ satisfies the conclusion of the Nash-Williams's Theorem.

Proof.

The proof goes by induction on the rank of the front. $\square$

Let $K \in K(2^{\mathbb{N}})$ be hereditary (closed under subsets) and tall. Define $\mathcal{F}_K = \{s \in [\mathbb{N}]^{<\mathbb{N}} : s \notin K\}$. It can be easily verified that for $\mathcal{F}_K$ is the first possibility of Galvin's Theorem never satisfied and that the set of all $\mathcal{F}_K$-monochromatic subsets is exactly K.

Theorem (G.-Uzcátegui)

There is $\mathcal{F} \subseteq [\mathbb{N}]^{<\mathbb{N}}$ such that there is no Borel function $S : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ such that $S(x) \subseteq x$ and $S(x)$ satisfies the conclusion of the Galvin's Theorem.

Proof.

Take some tall F_σ ideal $\mathcal{I}$ that does not have a selector. Pick any $K \in K(2^{\mathbb{N}})$ such that $\mathcal{I} = \mathcal{I}_K$ and K is hereditary. Then $\mathcal{F}_K$ works as desired. □

Thank you.