Basis and antibasis results for actions of locally compact groups

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Preliminaries

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The *induced orbit equivalence relation* E_G^X on X is given by $x \; E_G^X \; y$ if there is a $g \in G$ such that $g \cdot x = y$.

Motivation

Consider all continuous free actions of a locally compact group G on Polish spaces such that the induced orbit equivalence relation is non-smooth. Is there a small basis for these actions under G-embedibility? Similarly, is there such a basis for topologically weakly mixing actions, topologically strong mixing actions etc?

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- Let $\mathcal{F}$ be a subset of $\mathcal{P}(G^d)$. Then a continuous action of G on X is $\mathcal{F}$-recurrent if $\Delta(U, U)^d \in \mathcal{F}$ for all non-empty open subsets U of X.

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- If $\mathcal{S}$ is a subset of $\mathcal{P}(G^d)$, define $\mathcal{F}_{\mathcal{S}} = \{F \in \mathcal{P}(G^d) \mid \forall S \in \mathcal{S}\ F \cap S \neq \emptyset\}$.

Weakly mixing

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Let $D \subset G$ be a countable dense subgroup of G.
Let $\mathcal{S}$ be the family of subsets of G^2 of the form $S_{g,f} = \{(h, ghf) : h \in G)\}$ for $(g, f) \in D$.

Proposition

The action of G on X is weakly mixing iff it is $\mathcal{F}_{\mathcal{S}}$-recurrent and topologically transitive.

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If the action is topologically transitive and $\mathcal{F}_S$-recurrent, take non-empty open sets U_0, U_1, U_2, U_3 and find (by topological transitivity) a non-empty open set V and $f_i \in D$ for $i < 4$ such that $f_i V \subseteq U_i$ for $i < 4$.

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$\Delta(V, V)^2 \cap S_{f_1^{-1}f_3, f_2^{-1}f_0} \neq \emptyset$ shows that
$\Delta(f_0 V, f_1 V) \cap \Delta(f_1 V, f_2 V) \neq \emptyset$. □

Cocycle

- Let E be an equivalence relation on X. A *cocycle* ρ is a map from E to G such that $\rho(x,z) = \rho(x,y)\rho(y,z)$ for all $x\ E\ y\ E\ z$.
- For a cocycle $\rho : E \to G$ the equivalence relation E_ρ is defined by $x\ E_\rho\ y$ iff $x\ E\ y$ and $\rho(x,y) = 1$.

Definition

Suppose $d > 0$ and $\lambda : d \times \mathbb{N} \to G$. Define for $n \in \mathbb{N}$ and $s \in (d \times 2)^n$ $\lambda_s = \lambda_{(s(0)(0),0)}^{s(0)(1)} \ldots \lambda_{(s(n-1)(0),n-1)}^{s(n-1)(1)}$. Define a cocycle $\rho_\lambda : I_G \times E_0 | (d \times 2)^{\mathbb{N}} \to G$ by $\rho_\lambda((g, s^\frown x), (h, t^\frown x)) = g \lambda_s \lambda_t^{-1} h^{-1}$ for $s, t \in (d \times 2)^n$ and $x \in (d \times 2)^{\mathbb{N}}$.

Lemma

Suppose G is a locally compact Polish group, X is a locally compact Polish space, G acts continuously on X, E is a Borel equivalence relation containing E_X^G, $\rho : E \to G$ is a Borel cocycle such that $\rho(g \cdot x, x) = g$ for all $g \in G$ and $x \in X$. Suppose furthermore, that the E_ρ-saturation of every open set $U \subseteq X$ is open and that E_ρ is closed in $X \times X$. Than the quotient topology on X/E_ρ is locally compact and Polish and the induced action of G on X/E_ρ is continuous.

Theorem (Miller, I.)

Suppose that $d > 1$ and S is a countable set of subsets of G^d. Then for every topologically transitive and $\mathcal{F}_{\mathcal{S}}$-recurrent free action with non-smooth induced orbit equivalence relation, there is a sequence $\lambda : d \times \mathbb{N} \to G$ such that E_{ρ_λ} is closed, the action of G on $G \times (d \times 2)^{\mathbb{N}} / E_{\rho_\lambda}$ embeds into the action of G on X and is $\mathcal{F}$-transitive. Furthermore, there are continuum many pairwise incompatible actions with these properties.

Corollary (Miller, I.)

For every topologically weakly mixing action of G with no open orbit, there is a sequence $\lambda : 2 \times \mathbb{N} \to G$ such that E_{ρ_λ} is closed, the action of G on $G \times (2 \times 2)^{\mathbb{N}} / E_{\rho_\lambda}$ embeds into the action of G on X and is topologically weakly mixing. Furthermore, there are continuum many pairwise incompatible actions with these properties. In addition, if G is abelian, then there is a regular and σ-finite measure on $G \times (2 \times 2)^{\mathbb{N}} / E_{\rho_\lambda}$ for which the action of G is weakly mixing.

Other mixing notions

- The action of G on X is *topologically mixing* if it is topological transitive and $\mathcal{F}_{mixing}$-recurrent, where $\mathcal{F}_{mixing} = \{F \subseteq G : F^c \text{ is precompact}\}$.
- X is mildly mixing if every diagonal product action with a topological transitive action is again topological transitive.

Thank you!