The Banach–Mazur game and the strong Choquet game in domain theory

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A partially ordered set $(P, \sqsubseteq)$ is a dcpo (directed complete), if every directed subset $D \subseteq P$ has a least upper bound, denoted by $\bigsqcup D$.
In a poset $(P, \sqsubseteq)$ $a \ll b$ ("a is approximates b") if for each directed set $D \subseteq P$

$$b \sqsubseteq \bigsqcup D \Rightarrow \exists (d \in D)\ a \sqsubseteq d.$$

A dcpo P is said to be continuous if $\twoheaddownarrow(a) = \{b \in P : b \ll a\}$ is directed and has $a = \bigsqcup(\twoheaddownarrow(a))$ for each $a \in P$.
A domain is continuous dcpo.
A subset U of a poset P is Scott-open if

- U is an upper set: $x \in U$ and $x \sqsubseteq y$ then $y \in U$,
- for every directed $D \subseteq P$ which has a supremum,

$$\bigsqcup D \in U \Rightarrow D \cap U \neq \emptyset$$

Domains were discovered in computer science by D. Scott in 1970.

Domain theory

When a space X is homeomorphic to the space $\max(P)$ for a domain $(P, \sqsubseteq)$ with Scott topology inherited from P, Martin writes that X has a model, while Bennett and Lutzer write that X is domain representable.

- K. Martin, "Topological games in domain theory", 2003
- H. Benett, D. Lutzer, "Strong completness Properties in Topology", 2009

$\mathbb{R}$ is domain representable

$$P = \{[a,b] : a \leqslant b\}$$

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

$$[a,b] \ll [c,d] \Leftrightarrow [c,d] \subseteq (a,b)$$

$$\bigsqcup D = \bigcap D$$

for any directed set $D \subseteq P$

$$\max P = \{[x,x] : x \in \mathbb{R}\}$$

and $h\colon \max P \to \mathbb{R}$:

$$h([x,x]) = x$$

A locally compact Hausdorff space X is domain representable

$$P = \{K \subseteq X : \emptyset \neq K \text{ is compact}\}$$

$$K_1 \sqsubseteq K_2 \Leftrightarrow K_2 \subseteq K_1$$

$$K_1 \ll K_2 \Leftrightarrow K_2 \subseteq \operatorname{int} K_1$$

$$\bigsqcup D = \bigcap D$$

for any directed set $D \subseteq P$

$$\max P = \{\{x\} : x \in X\}$$

and $h\colon \max P \to X$

$$h(\{x\}) = x$$

- W. Fleissner, L. Yengulalp, "When $Cp(X)$ is Domain Representable", 2013

F-Y domain representable space

We say that a topological space X is F-Y (Fleissner–Yengulalp) countably domain representable if there is a triple $(Q, \ll, B)$ such that

(D1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a base for $\tau(X)$,

(D2) $\ll$ is a transitive relation on Q,

(D3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,

(D4) For all $x \in X$, a set $\{q \in Q : x \in B(q)\}$ is directed by $\ll$,

($D5_{\omega_1}$) if $D \subseteq Q$ and $(D, \ll)$ is countable and directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

If the conditions (D1)–(D4) and a condition

(D5) if $D \subseteq Q$ and $(D, \ll)$ is directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$

are satisfied, we say that a space X is F-Y domain representable.

Theorem[Martin, 2003]

A metric space is a domain reresentable iff it is completely metrizable.

Theorem[Benett, Lutzer, 2006]

If a space is Čech complete, then it is domain representable.

Theorem[Benett, Lutzer, 2006]

If a space X is domain representable and a space Y is a G_δ–subspace of X, then Y is a domain representable space.

F-Y π–domain representable space

We say that a topological space X is F-Y (Fleissner–Yengulalp) countably π-domain representable if there is a triple $(Q, \ll, B)$ such that

(πD1) $B : Q \to \tau^*(X)$ and $\{B(q) : q \in Q\}$ is a π-base for $\tau(X)$,

(πD2) $\ll$ is a transitive relation on Q,

(πD3) for all $p, q \in Q$, $p \ll q$ implies $B(p) \supseteq B(q)$,

(πD4) if $q, p \in Q$ satisfy $B(q) \cap B(p) \neq \emptyset$, there exists $r \in Q$ satisfying $p, q \ll r$,

(πD5_{ω_1}) if $D \subseteq Q$ and $(D, \ll)$ is countable and directed, then $\bigcap\{B(q) : q \in D\} \neq \emptyset$.

If the conditions (πD1)–(πD4) and a condition

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There exists a space, which it is F-Y countably domain representable (F-Y countably π-domain representable) but it is not F-Y π-domain representable (not F-Y domain representable.)
We consider a space

$$X = \Sigma\big(\{0,1\}^{\omega_1}\big) = \big\{x \in \{0,1\}^{\omega_1} : |\operatorname{supp} x| \leqslant \omega\big\},$$

where $\operatorname{supp} x = \{\alpha \in A : x(\alpha) = 1\}$ for $x \in \{0,1\}^A$, with the topology (ω_1-box topology) generated by a base

$$\mathcal{B} = \{\, \operatorname{pr}_A^{-1}(x) : A \in [\omega_1]^{\leqslant\omega}, x \in \{0,1\}^A\},$$

where $\operatorname{pr}_A : \Sigma(\{0,1\}^{\omega_1}) \to \{0,1\}^A$ is a projection.

$$Q = \mathcal{B} = \{\, \mathrm{pr}_A^{-1}(x) : A \in [\omega_1]^{\leqslant\omega}, x \in \{0,1\}^A \},$$

$$B : Q \to Q$$

be the identity.
A relation $\ll$ is defined in the following way

$$\mathrm{pr}_A^{-1}(x) \ll \mathrm{pr}_B^{-1}(y) \Leftrightarrow \mathrm{pr}_A^{-1}(x) \supseteq \mathrm{pr}_B^{-1}(y),$$

for any $\mathrm{pr}_A^{-1}(x), \mathrm{pr}_B^{-1}(y) \in \mathcal{B}$.

Two players α and β alternately choose open nonempty sets with

β	U_0		U_1		
					$\dots$
α		V_0		V_1	

Player α wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise β wins. Denoted this game by $BM(X)$.

Two players α and β alternately choose

β	$U_0 \ni x_0$		$U_1 \ni x_1$		
					$\dots$
α		V_0		V_1	

Player α wins if $\bigcap\{V_n : n \in \omega\} \neq \emptyset$. Otherwise β wins.

A strategy and a winning strategy

A strategy for the player α in the game $BM(X)$ or $Ch(X)$ is a rule for choosing what to play on each round given the full information of moves up until that point.
A winning strategy for the palyer α is a strategy that produces a win for that player α in any game when playing according to that startegy.

Theorem[Martin, 2003]

If a space X is domain representable, then the player α has a winning strategy in $Ch(X)$.

Theorem[Fleissner, Yengulalp, 2015]

If a space X is F-Y countably domain representable, then the player α has a winning strategy in $Ch(X)$.

Main result

Theorem[J.Bąk, A. K.]

If the player α has a winning strategy in $Ch(X)$, then X is F-Y countably domain representable.

Theorem[J.Bąk, A. K.]

The player α has a winning strategy in the BM(X) iff X is F-Y countably π- domain representable.

Thank You for Your attention!