Finite big Ramsey degrees in countable universal structures

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Big Ramsey degrees

$\mathcal{U}$ — a countably infinite structure

$\mathcal{A}$ — a finite structure

Definition. $T \in \mathbb{N}$ is a big Ramsey degree of $\mathcal{A}$ in $\mathcal{U}$ if for all $k \geqslant 2$ we have that $\mathcal{U} \longrightarrow (\mathcal{U})^{\mathcal{A}}_{k,T}$.

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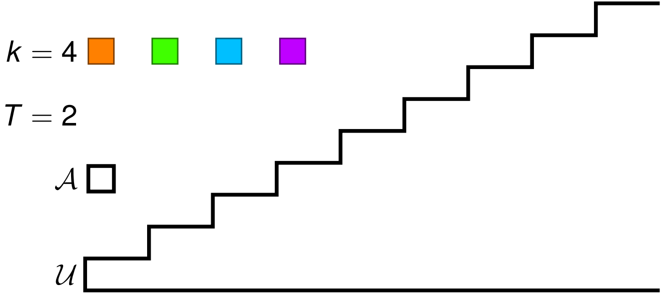

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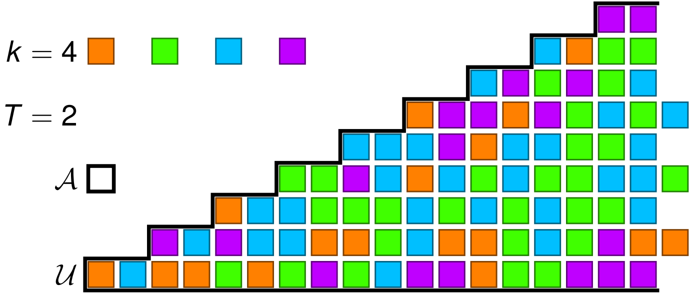

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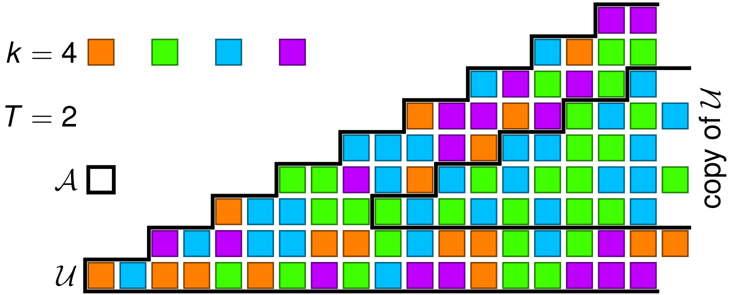

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$\mathcal{U} \longrightarrow (\mathcal{U})^{\mathcal{A}}_{k,T}$ in this talk:

For every coloring $\chi : \mathrm{Emb}(\mathcal{A}, \mathcal{U}) \to k$
there is a $w \in \mathrm{Emb}(\mathcal{U}, \mathcal{U})$
such that $|\chi(w \circ \mathrm{Emb}(\mathcal{A}, \mathcal{U}))| \leqslant T$

NB. Coloring embeddings VS coloring substructures

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$$T(\mathcal{A},\mathcal{U}) = \begin{cases} T, & \text{if such an integer exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

Big Ramsey degrees

Finite big Ramsey degrees in some Fraïssé limits:

- Finite chains ... in $\mathbb{Q}$
  [Devlin 1979]
- Finite graphs ... in the Rado graph $\mathcal{R}$
  [Sauer 2006]
- Finite S-ultrametric spaces ... in the "Urysohn space" $\mathcal{Y}_S$
  [Van Thé 2008]
- Finite local orders ... in $\mathcal{S}(2)$
  [Laflamme, Van Thé, Sauer 2010]
- Finite triangle-free graphs ... in the Henson graph $\mathcal{H}_3$
  [PREVIOUS LECTURE]

Big Ramsey degrees

In this talk: focus on structures which are *not* Fraïssé limits.

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Why?

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Why?

Ramsey's Theorem.
For all $m \geqslant 1$ *and* $k \geqslant 2$ *and for every coloring* $\chi : [\omega]^m \to k$ *there is an infinite* $S \subseteq \omega$ *such that* $|\chi([S]^m)| = 1$.

In other words,

$$T(m, \omega) = 1, \quad \text{for all } m \geqslant 1.$$

Frank P. Ramsey

Image courtesy of Wikipedia

Big Ramsey degrees

In this talk: focus on structures which are *not* Fraïssé limits.

Why?

For all $m \geqslant 1$:

$$T(m,\omega) = 1 \qquad \text{VS} \qquad T(m,\mathbb{Q}) = \left(\frac{d}{dx}\right)^{2m-1} \text{tg}(0)$$

A piggyback result

Theorem. *Assume the following:*

- $\mathcal{F}$ *is a countably infinite relational structure*
- $\mathrm{Age}(\mathcal{F})$ *has the strong amalgamation property*
- $\mathbf{K} = \{(\mathcal{A}, \prec) : \mathcal{A} \in \mathrm{Age}(\mathcal{F})$ *and* $\prec$ *is a linear order on* A *such that* $(A, \prec)$ *is finite or has order type* $\omega\}$
- $\sqsubset$ *is a linear order on* F *of order type* ω*.*

Then:

- $\mathrm{Age}(\mathcal{F}, \sqsubset) = \mathbf{K}$*.*
- *If* $\mathcal{A}$ *has finite big Ramsey degree in* $\mathcal{F}$ *then* $(\mathcal{A}, \prec)$ *has finite big Ramsey degree in* $(\mathcal{F}, \sqsubset)$*.*

A piggyback result

Corollary.

- *Every finite linearly ordered graph has finite big Ramsey degree in $(\mathcal{R}, \sqsubset)$, where $\sqsubset$ is a linear order on R of order type ω.*
- *Every finite permutation has finite big Ramsey degree in the permutation $(\mathbb{Q}, <, \sqsubset)$, where $<$ is the usual ordering of the rationals and $\sqsubset$ is a linear order on $\mathbb{Q}$ of order type ω.*

NB. A *linearly ordered graph* is a pair $(G, <)$ where G is a graph and $<$ is a linear order on the vertices of G.

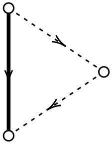
NB. A *permutation* is a structure $(A, <_1, <_2)$ where $<_1$ and $<_2$ are linear orders on A.

Acyclic digraphs

Theorem. *There exists a countably infinite acyclic digraph* $\mathcal{D}$ *such that every finite acyclic digraph* $\mathcal{A}$ *has finite big Ramsey degree in* $\mathcal{D}$.

A class of posets

Theorem. *Let* **K** *be a class of all finite linearly ordered posets which omit*

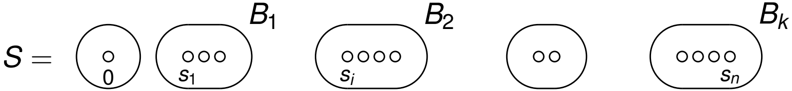


There exists a countably infinite linearly ordered poset $\mathcal{P}$ *such that every* $\mathcal{A} \in$ **K** *has finite big Ramsey degree in* $\mathcal{P}$*.*

NB. A linearly ordered poset $= (A, \leqslant, \prec)$ where $(A, \leqslant)$ is a poset and $\prec$ is a linear extension of $\leqslant$.

A class of metric spaces

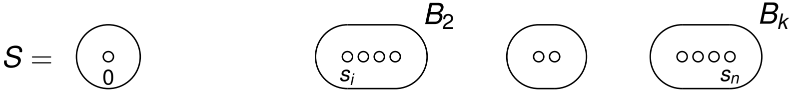
Every finite distance set $S = \{0 = s_0 < s_1 < \cdots < s_n\} \subseteq \mathbb{R}$ splits naturally into *blocks*:



Definition. A distance set S is compact if

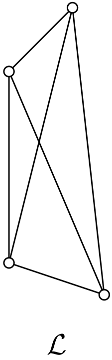
$$x \approx y \Leftrightarrow |x - y| \leqslant s_1, \quad \text{for all } x, y \in S.$$

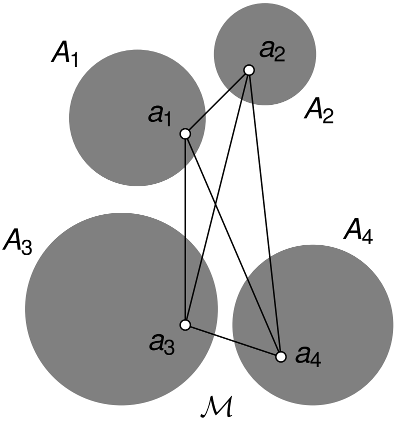
Let $S^+ = S \setminus B_1$:



A class of metric spaces

Definition. A finite S^+-metric space $\mathcal{L}$ *spans* an S-metric space $\mathcal{M}$ if the partition $\{A_1, A_2, \ldots, A_m\}$ of M into B_1-balls has a transversal $a_1 \in A_1$, $a_2 \in A_2$, …, $a_m \in A_m$ such that $\mathcal{M}\restriction_{\{a_1,a_2,\ldots,a_m\}} \cong \mathcal{L}$.





A class of metric spaces

Theorem. *Let*

- *S be a compact distance set,*
- *$\mathcal{L}$ be a finite S^+-metric space, and*
- *$\mathbf{K}_{S,\mathcal{L}}$ be the class of all finite S-met. spaces spanned by $\mathcal{L}$.*

Then there exists a countably infinite S-metric space $\mathcal{U}_{S,\mathcal{L}}$ such that every $\mathcal{M} \in \mathbf{K}_{S,\mathcal{L}}$ has finite big Ramsey degree in $\mathcal{U}_{S,\mathcal{L}}$.

In other words,

Every finite S-metric space $\mathcal{M}$ has finite big Ramsey degree in $\mathcal{U}_{S,\mathcal{L}}$ for every S^+-metric space $\mathcal{L}$ which spans $\mathcal{M}$.

Tools

Let $\mathbb{C}$ be a category and $A, B, C \in \mathrm{Ob}(\mathbb{C})$.

$C \longrightarrow (B)^A_k$ if:

for every mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

A category $\mathbb{C}$ has the Ramsey property if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A, B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ satisfying $C \longrightarrow (B)^A_k$

Tools

Let $\mathbb{C}$ be a category and $A, B, C \in \mathrm{Ob}(\mathbb{C})$.

$C \longrightarrow (B)^A_k$ if:

for every mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

$T \in \mathbb{N}$ is a big Ramsey degree of A in U if:

for all $k \geqslant 2$ we have that $U \longrightarrow (U)^A_{k,T}$.

Tools

Theorem. *Let*

- $\mathbb{B}$ *and* $\mathbb{C}$ *be categories,*
- $B \in \mathrm{Ob}(\mathbb{B})$ *and* $C \in \mathrm{Ob}(\mathbb{C})$*,*
- *there is a forgetful functor* $U : \overline{\mathrm{Age}}_{\mathbb{B}}(B) \to \overline{\mathrm{Age}}_{\mathbb{C}}(C)$*,*
- $U(B) = C$*;*
- *if* $U(B') = C$ *then* $\hom_{\mathbb{B}}(B, B') \neq \varnothing$*; and*
- *for every* $f \in \hom_{\mathbb{C}}(C, C)$ *there is a* $B' \in \mathrm{Ob}(\mathbb{B})$ *such that* $U(B') = C$ *and* $f \in \hom_{\mathbb{B}}(B', B)$*.*

Then $T_{\mathbb{B}}(A, B) \leqslant T_{\mathbb{C}}(U(A), C)$ *for all* $A \in \overline{\mathrm{Age}}_{\mathbb{B}}(B)$*.*

NB. $\overline{\mathrm{Age}}_{\mathbb{C}}(C) = \{A \in \mathrm{Ob}(\mathbb{C}) : \hom(A, C) \neq \varnothing\}$

Tools

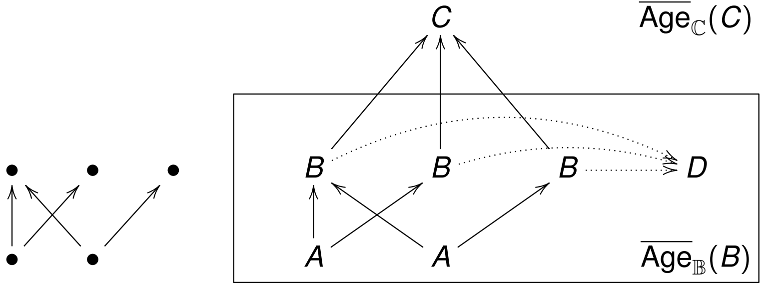
Theorem. *Let*

- $\mathbb{C}$ *be a category whose every morphism is monic,*
- $\mathbb{B}$ *be a (not necessarily full) subcategory of* $\mathbb{C}$*,*
- $B \in \mathrm{Ob}(\mathbb{B})$ *and* $C \in \mathrm{Ob}(\mathbb{C})$ *be such that* $\hom_{\mathbb{C}}(B, C) \neq \varnothing$*,*
- $A \in \overline{\mathrm{Age}}_{\mathbb{B}}(B)$
- *for every* (A, B)*-diagram* $F : \Delta \to \overline{\mathrm{Age}}_{\mathbb{B}}(B)$ *the following holds: if* F *has a commuting cocone in* $\overline{\mathrm{Age}}_{\mathbb{C}}(C)$ *whose tip is* C*, then* F *has a commuting cocone in* $\overline{\mathrm{Age}}_{\mathbb{B}}(B)$*.*

Then $T_{\mathbb{B}}(A, B) \leqslant T_{\mathbb{C}}(A, C)$*.*

Tools

Theorem. *In other words,*



$$\Delta \xrightarrow{F} \overline{\mathrm{Age}}_{\mathbb{B}}(B)$$

Future work

(jointly with Branislav Šobot)

Finite chains have finite big Ramsey deg's both in ω and in $\mathbb{Q}$.

Question: What other countable chains $\mathcal{C}$ have the property that $T(m, \mathcal{C}) < \infty$ for all m?

Future work

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Finite chains have finite big Ramsey deg's both in ω and in $\mathbb{Q}$.

Question: What other countable chains $\mathcal{C}$ have the property that $T(m, \mathcal{C}) < \infty$ for all m?

Fact. *If $\mathcal{U}$ and $\mathcal{V}$ are emb-equivalent structures then $T(\mathcal{A}, \mathcal{U}) = T(\mathcal{A}, \mathcal{V})$ for all finite $\mathcal{A} \in \mathrm{Age}(\mathcal{U}) = \mathrm{Age}(\mathcal{V})$.*

Non-scattered countable chains ✓

NB. $\mathcal{U}$ and $\mathcal{V}$ are emb-equivalent if $\mathcal{U} \hookrightarrow \mathcal{V}$ and $\mathcal{V} \hookrightarrow \mathcal{U}$.

Future work

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Finite chains have finite big Ramsey deg's both in ω and in $\mathbb{Q}$.

Question: What other countable chains $\mathcal{C}$ have the property that $T(m, \mathcal{C}) < \infty$ for all m?

Goal: Prove that scattered countable chains also have the property.