Cardinal invariants above the continuum

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Outline

Some cardinal invariants at regular cardinals

Definition

Let $\kappa \geq \omega$ be a regular cardinal. Let $f, g \in \kappa^{\kappa}$. $f \leq^{*} g$ means that $|\{\alpha < \kappa : g(\alpha) < f(\alpha)\}| < \kappa$

Definition

We say that $F \subseteq \kappa^{\kappa}$ is ***$*$-unbounded*** *if $\neg\exists g \in \kappa^{\kappa} \forall f \in F\, [f \leq^{*} g]$.*

Definition

$\mathfrak{b}(\kappa) = \min\{|F| : F \subseteq \kappa^{\kappa} \wedge F$ *is* $*$ *-unbounded*$\}$.

Definition

*We say that $F \subseteq \kappa^\kappa$ is $*$-**dominating** if $\forall g \in \kappa^\kappa \exists f \in F \, [g \leq^* f]$*

Definition

$\mathfrak{d}(\kappa) = \min \{|F| : F \subseteq \kappa^\kappa$ *and F is $*$-dominating*$\}$.

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$\mathfrak{d}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa$ *and* F *is* $*$ *-dominating*$\}$.

Theorem

For any regular $\kappa \geq \omega$, $\kappa^+ \leq \mathrm{cf}(\mathfrak{b}(\kappa)) = \mathfrak{b}(\kappa) \leq \mathrm{cf}(\mathfrak{d}(\kappa)) \leq \mathfrak{d}(\kappa) \leq 2^\kappa$

- These are the only relations between $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ that are provable in ZFC (Hechler for ω; Cummings and Shelah for $\kappa > \omega$).

- When $\kappa > \omega$, we can also use the club filter.

Definition

Let $\kappa > \omega$ be a regular cardinal. $f \leq_{\mathrm{cl}} g$ means that $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary. For $F \subseteq \kappa^\kappa$, we say that:

- *F is cl-**unbounded** if $\neg\exists g \in \kappa^\kappa \forall f \in F\, [f \leq_{\mathrm{cl}} g]$, and*
- *F is cl-**dominating** if $\forall g \in \kappa^\kappa \exists f \in F\, [g \leq_{\mathrm{cl}} f]$.*

Definition

We define

$$\mathfrak{b}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \wedge F \text{ is cl-unbounded}\},$$
$$\mathfrak{d}_{\mathrm{cl}}(\kappa) = \min\{|F| : F \subseteq \kappa^\kappa \text{ and } F \text{ is cl-dominating}\}.$$

Theorem (Cummings and Shelah)

For every regular cardinal $\kappa > \omega$, $\mathfrak{b}(\kappa) = \mathfrak{b}_{\mathrm{cl}}(\kappa)$.

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If $\kappa \geq \beth_{\omega}$ *is regular, then* $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$.

Question

Does $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$*, for every regular uncountable* κ*? In particular, does* $\mathfrak{d}(\omega_1) = \mathfrak{d}_{\mathrm{cl}}(\omega_1)$*?*

Definition

Let $\kappa \geq \omega$ be regular.

- *For $A, B \in \mathcal{P}(\kappa)$, A **splits** B if $|B \cap A| = |B \cap (\kappa \setminus A)| = \kappa$.*
- *$F \subseteq \mathcal{P}(\kappa)$ is called a **splitting family** if $\forall B \in [\kappa]^{\kappa} \exists A \in F \, [A \text{ splits} B]$.*

$$\mathfrak{s}(\kappa) = \min\{|F| : F \subseteq \mathcal{P}(\kappa) \wedge F \text{ is a splitting family}\};$$

Theorem (Solomon)

$\omega_1 \leq \mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$.

Theorem (Suzuki)

For a regular $\kappa > \omega$, $\mathfrak{s}(\kappa) \geq \kappa$ iff κ is strongly inaccessible and $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact.

- So if κ is not weakly compact, then $\mathfrak{s}(\kappa) < \kappa^+ \leq \mathfrak{b}(\kappa)$.

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Theorem (Zapletal)

If it is consistent to have a regular uncountable cardinal κ such that $\mathfrak{s}(\kappa) \geq \kappa^{++}$, then it is also consistent that there is a κ with $o(\kappa) \geq \kappa^{++}$.

Theorem (Ben-Neria and Gitik)

If $o(\kappa) = \kappa^{++}$, then there is a forcing extension in which $\mathfrak{s}(\kappa) = \kappa^{++}$.

- κ does not remain measurable in their model.

Question

What is the consistency strength of the statement that κ is a measurable cardinal and $\mathfrak{s}(\kappa) = \kappa^{++}$?

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- $\mathfrak{s}(\omega)$ and $\mathfrak{b}(\omega)$ are independent.

Theorem (Baumgartner and Dordal)

It is consistent to have $\mathfrak{s}(\omega) < \mathfrak{b}(\omega)$.

Theorem (Shelah)

It is consistent to have $\omega_1 = \mathfrak{b}(\omega) < \mathfrak{s}(\omega) = \omega_2$.

- Historically, Shelah's result was the first published use of creature forcing.

- It turns out the ω is the **only regular cardinal** for which the statement $\mathfrak{b}(\kappa) < \mathfrak{s}(\kappa)$ is consistent.

Theorem (R. and Shelah[2])

For any regular uncountable cardinal κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

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Theorem (R. and Shelah[2])

For any regular uncountable cardinal κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

- $\mathfrak{b}(\omega)$ and $\mathfrak{d}(\omega)$ are dual to each other
- The dual of $\mathfrak{s}(\omega)$ is $\mathfrak{r}(\omega)$.

Definition

*For a family $F \subseteq [\kappa]^{\kappa}$ and a set $B \in \mathcal{P}(\kappa)$, B is said to **reap** F if for every $A \in F$, $|A \cap B| = |A \cap (\kappa \setminus B)| = \kappa$. We say that $F \subseteq [\kappa]^{\kappa}$ is **unreaped** if there is no $B \in \mathcal{P}(\kappa)$ that reaps F.*

- $F \subseteq [\kappa]^\kappa$ is unreaped iff each $B \in \mathcal{P}(\kappa)$ is decided by some member of F.

Definition

$\mathfrak{r}(\kappa) = \min\{|F| : F \subseteq [\kappa]^\kappa$ *and* F *is unreaped*$\}$.

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Definition

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- The proof of $\mathfrak{s}(\omega) \leq \mathfrak{d}(\omega)$ dualizes to the proof of $\mathfrak{b}(\omega) \leq \mathfrak{r}(\omega)$.
- Also $\mathfrak{r}(\omega)$ and $\mathfrak{d}(\omega)$ are independent.
- Not clear if there is a good theory of duality at uncountable regular cardinals too.
- For example, Suzuki's theorem says that $\mathfrak{s}(\kappa)$ is small unless κ is weakly compact.
- So we might expect that $\mathfrak{r}(\kappa)$ is large below the first weakly compact cardinal.

Question

Is it consistent (relative to large cardinals) that there is some uncountable regular cardinal κ below the first weakly compact cardinal such that $\mathfrak{r}(\kappa) < 2^{\kappa}$?

- My conjecture is yes (so Suzuki's theorem has no dual).

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- But the theorem does have a partial dual:

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Theorem (R. + Shelah [3])

for all regular cardinals $\kappa \geq \beth_{\omega}$, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

- So for sufficiently large κ, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ provably in ZFC.

Question

Is $\mathfrak{d}(\aleph_1) \leq \mathfrak{r}(\aleph_1)$ provable? Is $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ provable for all regular $\kappa < \beth_\omega$?

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Question

Is it consistent (relative to large cardinals) that $\mathfrak{r}(\omega_1) < 2^{\aleph_1}$?

- This is related to an old question of Kunen about bases for uniform ultrafilters.

Definition

Let $\kappa \geq \omega$ be regular. Let $\mathcal{U}$ be an ultrafilter on κ. We say that:

- *$\mathcal{U}$ is **uniform** if every element of $\mathcal{U}$ has cardinality κ;*
- *$F \subseteq \mathcal{P}(\kappa)$ is a **base for** $\mathcal{U}$ if $\mathcal{U} = \{B \subseteq \kappa : \exists A \in F\,[A \subseteq B]\}$.*

Definition

$\mathfrak{u}(\kappa) = \min\{|F| : F$ *is a base for a uniform ultrafilter on* $\kappa\}$.

- Clearly $\mathfrak{r}(\kappa) \leq \mathfrak{u}(\kappa)$.
- $\mathfrak{u}(\omega)$ and $\mathfrak{s}(\omega)$ are independent.
- However for $\kappa > \omega$, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$.

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- However for $\kappa > \omega$, $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa) \leq \mathfrak{r}(\kappa)$.

Question (Kunen)

Is it consistent that $\mathfrak{u}(\omega_1) < 2^{\aleph_1}$ *?*

Theorem (Garti and Shelah)

If κ *is supercompact, then* $\mathfrak{u}(\kappa) < 2^{\kappa}$ *is consistent.*

Definition

Let $\kappa \geq \omega$ be a regular cardinal.

- $A, B \in [\kappa]^\kappa$ *are said to be* ***almost disjoint*** *or* ***a.d.*** *if* $|A \cap B| < \kappa$.
- *A family* $\mathscr{A} \subseteq [\kappa]^\kappa$ *is said to be* ***almost disjoint*** *or* ***a.d.*** *if the members of* $\mathscr{A}$ *are pairwise a.d.*
- *Finally* $\mathscr{A} \subseteq [\kappa]^\kappa$ *is called* ***maximal almost disjoint*** *or* ***m.a.d.*** *if* $\mathscr{A}$ *is an a.d. family,* $|\mathscr{A}| \geq \kappa$*, and* $\mathscr{A}$ *cannot be extended to a larger a.d. family in* $[\kappa]^\kappa$.

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- *Finally $\mathscr{A} \subseteq [\kappa]^{\kappa}$ is called **maximal almost disjoint** or **m.a.d.** if $\mathscr{A}$ is an a.d. family, $|\mathscr{A}| \geq \kappa$, and $\mathscr{A}$ cannot be extended to a larger a.d. family in $[\kappa]^{\kappa}$.*

Definition

$\mathfrak{a}(\kappa) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq [\kappa]^{\kappa}$ *and* $\mathscr{A}$ *is m.a.d.*$\}$.

Theorem (Rothberger)

For any regular $\kappa \geq \omega$, $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa)$.

Theorem (Shelah)

It is consistent to have $\aleph_1 = \mathfrak{b}(\omega) < \mathfrak{a}(\omega) = \aleph_2 = \mathfrak{s}(\omega)$. *It is also consistent to have* $\aleph_1 = \mathfrak{b}(\omega) = \mathfrak{a}(\omega) < \mathfrak{s}(\omega)$.

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- It turns out that ω is the *only regular* κ where $\mathfrak{b}(\kappa) = \kappa^+ < \kappa^{++} = \mathfrak{a}(\kappa)$ is consistent.

Theorem (R. + Shelah)

If $\kappa > \omega$ *is regular, then* $\mathfrak{b}(\kappa) = \kappa^+$ *implies* $\mathfrak{a}(\kappa) = \kappa^+$.

Theorem (Blass, Hyttinen, and Zhang)

Let $\kappa > \omega$ be regular. If $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.

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Let $\kappa > \omega$ be regular. If $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.

Question (Roitman)

Does $\mathfrak{d}(\omega) = \aleph_1$ imply that $\mathfrak{a}(\omega) = \aleph_1$?

Theorem (Shelah)

It is consistent to have $\aleph_2 = \mathfrak{d}(\omega) < \mathfrak{a}(\omega) = \aleph_3$.

- He actually gave two different proofs of $\mathrm{Con}(\mathfrak{d}(\omega) < \mathfrak{a}(\omega))$.
- The first proof used ultrapowers and needed a measurable cardinal θ to produce a model with $\theta < \mathfrak{d}(\omega) < \mathfrak{a}(\omega)$.
- The other proof used templates and produced a model with $\mathfrak{d}(\omega) = \aleph_2$.

- The first proof also works for $\mathfrak{u}(\omega)$.

Theorem (Shelah)

Suppose there is a measurable cardinal θ. Then there is a c.c.c. forcing extension in which $\theta < \mathfrak{u}(\omega) < \mathfrak{a}(\omega)$.

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Theorem (Shelah)

Suppose there is a measurable cardinal θ. Then there is a c.c.c. forcing extension in which $\theta < \mathfrak{u}(\omega) < \mathfrak{a}(\omega)$.

Question

What is the consistency strength of $\mathfrak{u}(\omega) < \mathfrak{a}(\omega)$?

Consistency results

- R. + Shelah used the method of Boolean ultrapowers to get several consistency results involving $\mathfrak{a}(\kappa)$.

Theorem (R. + Shelah [1])

For any regular $\kappa > \omega$, $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$ is consistent relative to a supercompact cardinal.

- This is analogous to Shelah's first result that $\mathfrak{d}(\omega) < \mathfrak{a}(\omega)$ is consistent relative to a measurable.
- The consistency of $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$ for uncountable κ was also unknown before this result.

Theorem

More specifically, suppose $\aleph_0 < \kappa = \kappa^{<\kappa} < \theta$ and that θ is supercompact. Then there is a forcing extension in which $\theta < \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$.

- We can also arrange $\mathfrak{b}(\kappa)$ and $\mathfrak{d}(\kappa)$ to be different.

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Suppose $\aleph_0 < \kappa = \kappa^{<\kappa} < \theta$ and that θ is supercompact. Then there is a forcing extension in which $\theta < \mathfrak{b}(\kappa) < \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$.

Question

What is the consistency strength of the statement that there is an uncountable regular cardinal κ for which $\mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$, or even $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$?

Question

For uncountable regular κ, does $\mathfrak{b}(\kappa) = \kappa^{++}$ imply that $\mathfrak{a}(\kappa) = \kappa^{++}$?

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Theorem (R. + Shelah [1])

If κ *is a Laver indestructible supercompact cardinal, then* $\mathfrak{u}(\kappa) < \mathfrak{a}(\kappa)$ *is consistent relative to a supercompact cardinal above* κ*. More specifically, suppose that* $\kappa < \theta$*, that* θ *is supercompact, and that* κ *is Laver indestructible supercompact. Then there is a forcing extension in which* $\theta < \mathfrak{u}(\kappa) < \mathfrak{a}(\kappa)$*.*

- This is analogous to Shelah's that $\theta < \mathfrak{u}(\omega) < \mathfrak{a}(\omega)$ is consistent if θ is measurable.

Definition

Suppose θ *supercompact,* $\theta \leq \mu = \mu^{<\theta} < \mu^+ < \chi$. $\mathcal{B}_{\chi,\mu,\theta}$ *is the completion of* $\mathrm{Fn}(\chi, \mu, \theta) = \{f : \mathrm{dom}(f) \in [\chi]^{<\theta}$ *and* $\mathrm{ran}(f) \subseteq \mu\}$ *ordered by reverse inclusion.*

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- Build a θ-complete **optimal** ultrafilter D on $\mathcal{B}_{\chi,\mu,\theta}$ (using the fact that θ is supercompact).
- For getting a model with $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$, fix the usual iteration $\mathbb{P}$ for forcing $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mu^+$.

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- Build a θ-complete **optimal** ultrafilter D on $\mathcal{B}_{\chi,\mu,\theta}$ (using the fact that θ is supercompact).
- For getting a model with $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) < \mathfrak{a}(\kappa)$, fix the usual iteration $\mathbb{P}$ for forcing $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mu^+$.
- Let $\mathbb{Q} = \mathbb{P}^{[\mathcal{B}_{\chi,\mu,\theta}]}/D$.
- Forcing with $\mathbb{Q}$ preserves $\mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mu^+$ and makes $\mathfrak{a}(\kappa) = \mathrm{cf}(\chi)$.

An application of PCF theory

Theorem

For any regular $\kappa \geq \beth_\omega$, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

Definition

Let $\kappa > \omega$ *be a regular cardinal. If* $A \in [\kappa]^\kappa$, *then we define a function* $s_A : \kappa \to A$ *by setting* $s_A(\alpha) = \min(A \setminus (\alpha + 1))$, *for each* $\alpha \in \kappa$.

Definition

Let $E_2 \subseteq E_1$ *both be clubs in* κ. *For each* $\xi \in \kappa$, *define* $\mathrm{set}\,(E_1, \xi) = \{\zeta < s_{E_1}(\xi) : \xi \leq \zeta\}$. *Define* $\mathrm{set}\,(E_2, E_1) = \bigcup \{\mathrm{set}\,(E_1, \xi) : \xi \in E_2\}$.

- Assume $\kappa \geq \beth_\omega$. Let $F \subseteq [\kappa]^\kappa$ be such that F is unreaped and $|F| = \mathfrak{r}(\kappa)$.
- We will need the revised GCH

Definition

Let κ and λ be cardinals. Define $\lambda^{[\kappa]}$ to be

$$\min\left\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\kappa} \text{ and } \forall u \in [\lambda]^\kappa \exists \mathcal{P}_0 \subseteq \mathcal{P}\left[|\mathcal{P}_0| < \kappa \text{ and } u = \bigcup \mathcal{P}_0\right]\right\}.$$

*The operation $\lambda^{[\kappa]}$ is sometimes referred to as the **weak power**.*

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*The operation $\lambda^{[\kappa]}$ is sometimes referred to as the **weak power**.*

- Easy exercise: GCH is equivalent to the statement that for all regular cardinals $\kappa < \lambda$, $\lambda^{[\kappa]} = \lambda$.
- The revised GCH, which is a theorem of ZFC says that for “lots of pairs” of regular cardinals we have $\lambda^{[\kappa]} = \lambda$.

Theorem (Shelah; The Revised GCH)

If θ is a strong limit uncountable cardinal, then for every $\lambda \geq \theta$, there exists $\sigma < \theta$ such that for every $\sigma \leq \kappa < \theta$, $\lambda^{[\kappa]} = \lambda$.

Corollary

Let $\mu \geq \beth_\omega$ be any cardinal. There exists an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\mu]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and for each $u \in [\mu]^{\theta}$, there exists $v \in \mathcal{P}$ with the property that $v \subseteq u$ and $|v| \geq \aleph_0$.

- Applying this with $\mu = \mathfrak{r}(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^{\theta}$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.

- Applying this with $\mu = \mathfrak{r}(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^{\theta}$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.
- Fix $M \prec H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.
- $M \cap \kappa^\kappa$ is a dominating family (this shows $\mathfrak{d}(\kappa) \leq \mu$).

- Applying this with $\mu = \mathfrak{r}(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^{\theta}$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.
- Fix $M \prec H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.
- $M \cap \kappa^\kappa$ is a dominating family (this shows $\mathfrak{d}(\kappa) \leq \mu$).
- It may be assumed that for any club $E_1 \subseteq \kappa$, there exists a club $E_2 \subseteq E_1$ such that for all $B \in F$, $B \not\subseteq^* \operatorname{set}(E_2, E_1)$ (otherwise there is an easy argument).

- Applying this with $\mu = \mathfrak{r}(\kappa)$, fix an uncountable regular cardinal $\theta < \beth_\omega$ and a family $\mathcal{P} \subseteq [\theta \times F]^{\leq\theta}$ such that $|\mathcal{P}| \leq \mu$ and $\mathcal{P}$ has the property that for each $u \in [\theta \times F]^\theta$, there exists $v \in \mathcal{P}$ satisfying $v \subseteq u$ and $|v| \geq \aleph_0$.
- Fix $M \prec H(\chi)$ containing everything relevant with $|M| = \mu$ and $F \subseteq M$.
- $M \cap \kappa^\kappa$ is a dominating family (this shows $\mathfrak{d}(\kappa) \leq \mu$).
- It may be assumed that for any club $E_1 \subseteq \kappa$, there exists a club $E_2 \subseteq E_1$ such that for all $B \in F$, $B \not\subseteq^* \operatorname{set}(E_2, E_1)$ (otherwise there is an easy argument).
- Since F is an unreaped family, it follows that for each club $E_1 \subseteq \kappa$, there exist a club $E_2 \subseteq E_1$ and a $B \in F$ such that $B \subseteq^* \kappa \setminus \operatorname{set}(E_2, E_1)$.

- Let $f \in \kappa^\kappa$ be a fixed function.
- Construct a sequence $\langle\langle E_i, E_i^1, B_i\rangle : i < \theta\rangle$ so that the following conditions are satisfied at each $i < \theta$:
  1. E_i and E_i^1 are both clubs in κ, $E_i^1 \subseteq E_i$, and $\forall j < i \left[E_i \subseteq E_j^1\right]$;
  2. $B_i \in F$ and $B_i \subseteq^* \kappa \setminus \text{set}\left(E_i^1, E_i\right)$;
  3. if $i = 0$, then $E_i = \{\alpha < \kappa : \alpha \text{ is closed under } f\}$.

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  2. $B_i \in F$ and $B_i \subseteq^* \kappa \setminus \operatorname{set}\left(E_i^1, E_i\right)$;
  3. if $i = 0$, then $E_i = \{\alpha < \kappa : \alpha$ is closed under $f\}$.
- define $u : \theta \to F$ by setting $u(i) = B_i$ for all $i \in \theta$.
- By the choice of $\mathcal{P}$ and M, we can find a sub-function $w \subseteq u$ in M so that $\operatorname{otp}(\operatorname{dom}(w)) = \omega$.
- Let $\langle i_n : n \in \omega\rangle$ be the strictly increasing enumeration of $\operatorname{dom}(w)$.

- By regularity of κ, there exists a function $g \in M \cap \kappa^\kappa$ with the property that for each $\alpha \in \kappa$, $\forall i \in \mathrm{dom}(w) \, [B_i \cap [\alpha, g(\alpha)) \neq 0]$.

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- Find $\delta < \kappa$ so that for each $n \in \omega$:
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- We will show that for any $\alpha > \delta$, $f(\alpha) < g(\alpha)$.

- Fix $\alpha > \delta$ and define $\xi_n = \sup(E_{i_n} \cap (\alpha + 1))$.
- Then $\xi_n \in E_{i_n}$ and they are non-increasing.
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- Fix $\beta \in B_{i_N} \cap [\alpha, g(\alpha))$.
- Then $\beta \notin \operatorname{set}\left(E^1_{i_N}, E_{i_N}\right)$.
- Note $\xi = \xi_{N+1} \in E_{i_{N+1}} \subseteq E^1_{i_N}$.
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- Hence $\beta \notin \left[\xi, s_{E_{i_N}}(\xi)\right)$.
- On the other hand, $\xi \leq \alpha \leq \beta$. Hence $\xi \leq \alpha < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$.
- Finally, since $s_{E_{i_N}}(\xi) \in E_{i_N} \subseteq E_0$, $s_{E_{i_N}}(\xi)$ is closed under f.
- Therefore, $f(\alpha) < s_{E_{i_N}}(\xi) \leq \beta < g(\alpha)$.

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