Relation between Ideal convergence and Sequence selection principle

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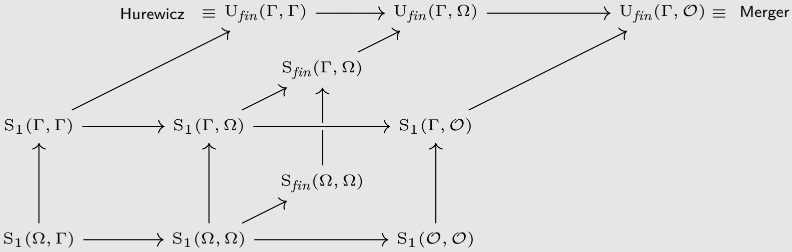
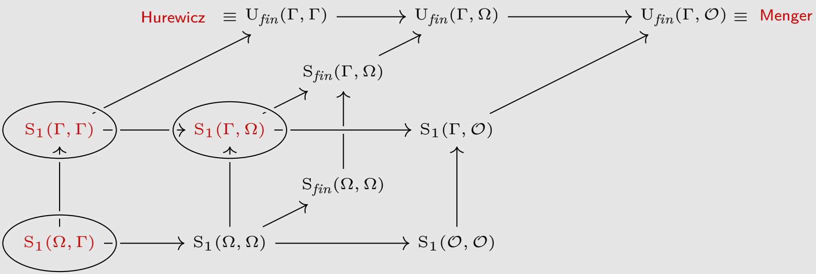


Diagram. Scheepers' diagram.



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Covering and ideals

- The family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is called **ideal**, if
  - it is closed under taking subsets and finite unions
  - does not contain the set ω, but contains all finite subsets of ω.
- E.g.: the Frechét ideal, denoted as Fin, is a set $[\omega]^{<\aleph_0}$.
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- the sequence $\langle U_n : n \in \omega \rangle$ of subsets of X is called an ω-**cover**, if for every $n \in \omega$, $U_n \neq X$ and for **every finite** $F \subseteq X$ there is n such that $F \subseteq U_n$, see [7].
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- the sequence $\langle U_n : n \in \omega \rangle$ of subsets of X is called an $\mathcal{I}$-γ-**cover**, if for every $n \in \omega$, $U_n \neq X$ and for every $x \in X$, the set $\{n \in \omega : x \notin U_n\} \in \mathcal{I}$, see [3].
  - $\mathcal{I}$-Γ denotes the family of all open $\mathcal{I}$-γ-covers of X.
  - $\mathrm{Fin}\text{-}\Gamma = \Gamma$.

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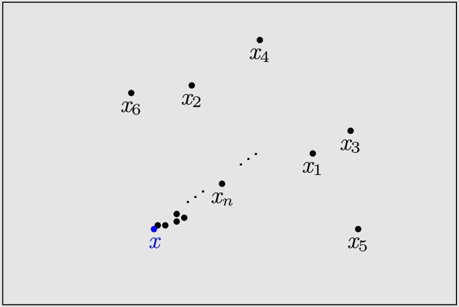

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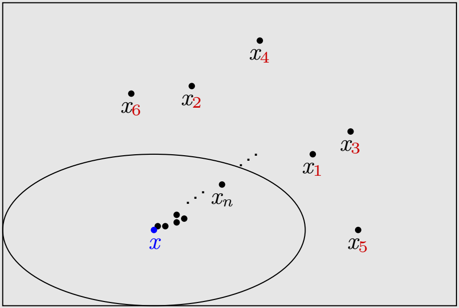

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- especially, if control sequence is $\langle 2^{-n} : \ n \in \omega \rangle$ we are talking about **strongly** $\mathcal{I}$-**quasi normal convergence** of f_n to f, written $f_n \xrightarrow{\mathrm{s}\mathcal{I}\mathrm{QN}} f$.

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- We use $\Gamma_{\mathbf{0}}$ instead of $\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}$.

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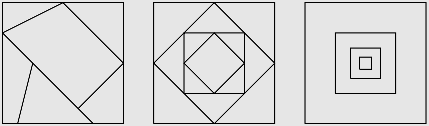

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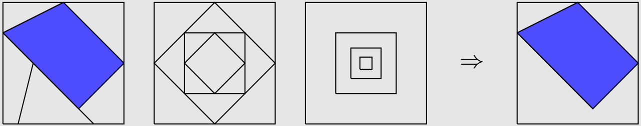

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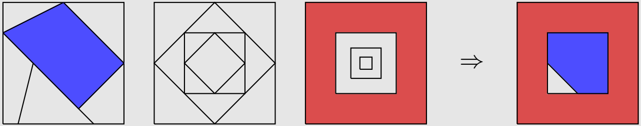

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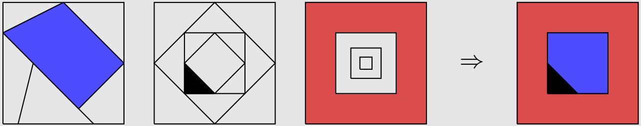

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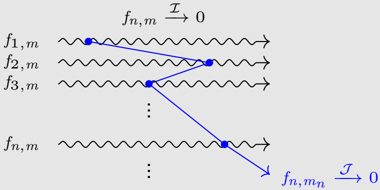
Let $\mathcal{I}, \mathcal{J}$ be ideals on ω. Then $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ can be imagined by follow way



Covering $\mathrm{S}_1(\mathcal{P},\mathcal{R})$ and $\mathrm{S}_1(\mathcal{P},\mathcal{R})$ for functions

Observation

(1) If X is an $\mathrm{S}_1(\Gamma,\mathcal{J}\text{-}\Gamma)$-space then X is an $\mathrm{S}_1(\Gamma,\Omega)$-space.

(2) If $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{0}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$-space then $\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$-space.

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Proposition (V.Š.,J.Šupina)

Let X be a topological space. Then

(1) *X is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)$-space if and only if X has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$ and $\mathrm{S}_1(\Gamma, \Gamma)$.*

(2) *$\mathrm{C}_p(X)$ is an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$-space if and only if $\mathrm{C}_p(X)$ has $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$ and $\mathrm{S}_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$.*

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Lemma (V.Š., J.Šupina)

(1) *For any countable ω-cover $\mathcal{U}$ of X and its bijective enumeration $\langle U_n : n\in\omega\rangle$ there is an ideal $\mathcal{I}$ such that $\langle U_n : n\in\omega\rangle$ is an $\mathcal{I}$-γ-cover.*

(2) *For any countable family of functions $\mathcal{E}$ on X such that $\mathbf{0}\in\overline{\mathcal{E}\setminus\{\mathbf{0}\}}$ and its bijective enumeration $\langle f_n : n\in\omega\rangle$ there is an ideal $\mathcal{I}$ such that $f_n \xrightarrow{\mathcal{I}} \mathbf{0}$.*

Covering $\mathrm{S}_1(\mathcal{P},\mathcal{R})$ and $\mathrm{S}_1(\mathcal{P},\mathcal{R})$ for functions

Let us recall a folklore result by J. Gerlits and Zs. Nagy [4] for a Tychonoff space X:

$$X \text{ has } \binom{\Omega}{\Gamma} \Leftrightarrow X \text{ has } \mathrm{S}_1(\Omega,\Gamma) \Leftrightarrow \mathrm{C}_p(X) \text{ has } \binom{\Omega_{\mathbf{0}}}{\Gamma_{\mathbf{0}}} \Leftrightarrow \mathrm{C}_p(X) \text{ has } \mathrm{S}_1(\Omega_{\mathbf{0}},\Gamma_{\mathbf{0}}).$$

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Theorem (V.Š.,J.Šupina)

Let X be a Tychonoff topological space. The following statements are equivalent.

(a) X *is an* $\mathrm{S}_1(\Omega,\Gamma)$*-space.*

(b) X *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma)$*-space for every ideal* $\mathcal{I}$*.*

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$$\begin{array}{ccccc}
\mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\mathcal{J}\text{-}\Gamma) & \longrightarrow & \mathrm{S}_1(\Gamma,\Omega) \\
\uparrow & & \uparrow & & \\
\mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1(\mathcal{I}\text{-}\Gamma,\mathcal{J}\text{-}\Gamma) & & \\
\uparrow & & & & \\
\mathrm{S}_1(\Omega,\Gamma) & & & &
\end{array}$$

Diagram. Covering selection principles.

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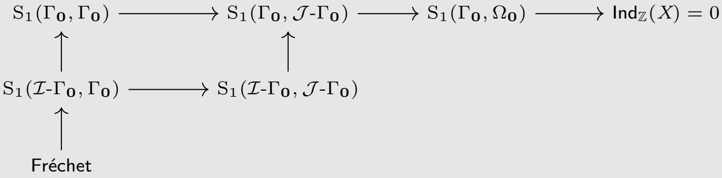

Diagram. Selection principles for functions.

Monotonne version of $\mathrm{S}_1(\mathcal{P}, \mathcal{R})$ for functions

- We say that a sequence $\langle f_n : \ n \in \omega \rangle$ is **monotone sequence** if for any $n \in \omega$ and $x \in X$ we have $f_n(x) \geq f_{n+1}(x)$.

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Lemma (V.Š., J.Šupina)

Let X be a topological space.

(1) $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^{\mathrm{m}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}^{\mathrm{m}},\mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

(2) $\mathrm{C}_p(X)$ *has the property* $\begin{bmatrix}\Gamma_{\mathbf{0}}^{\mathrm{m}}\\ \mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{bmatrix}$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\begin{bmatrix}\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}^{\mathrm{m}}\\ \mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{bmatrix}$.

(3) $\mathrm{C}_p(X)$ *has the property* $\begin{bmatrix}\Gamma_{\mathbf{0}}^{\mathrm{m}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{bmatrix}$ *if and only if* $\mathrm{C}_p(X)$ *has the property* $\begin{bmatrix}\mathrm{Fin}\text{-}\Gamma_{\mathbf{0}}^{m}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{bmatrix}$.

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Conection between coverings and functions

- We say that a topological space X has $\mathcal{J}$-**Hurewicz property** if for each sequence $\langle \mathcal{U}_n : \ n \in \omega \rangle$ of open covers of X there are finite $\mathcal{V}_n \subset \mathcal{U}_n$, $n \in \omega$ such that for each $x \in X$, $\{n \in \omega : \ x \notin \bigcup \mathcal{V}_n\} \in \mathcal{J}$.[3].

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Proposition (V.Š., J.Šupina)

If X is a perfectly normal topological space then the following are equivalent. Moreover, if X is arbitrary topological space then $(\mathrm{a}) \equiv (\mathrm{b})$.

(a) $\mathrm{C}_p(X)$ *has* $\left[{\Gamma^{\mathrm{m}}_{\mathbf{0}} \atop s\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]$.

(b) $\mathrm{C}_p(X)$ *has the property* $\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

(c) X *possesses a* $\mathcal{J}$*-Hurewicz property.*

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\end{array}$$

Diagram. Monotonic selection principles for functions.

Theorem (L. Bukovský, P. Das, J.Šupina.[1])

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω. If X is a normal topological space then the following are equivalent.
Moreover, the equivalence $(\mathrm{a}) \equiv (\mathrm{b})$ holds for arbitrary topological space X.

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(b) $\mathrm{C}_p(X)$ *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space.*

(c) X *is an* $S_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma)$*-space.*

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- As a corollary L. Bukovský, P. Das and J. Š. obtained the ideal version of Scheepers' result [9].

$$\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma) \rightarrow S_1(\mathcal{I}\text{-}\Gamma^{sh}, \mathcal{J}\text{-}\Gamma) \Leftrightarrow \mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}) \rightarrow \mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}}).$$

Conection between coverings and functions

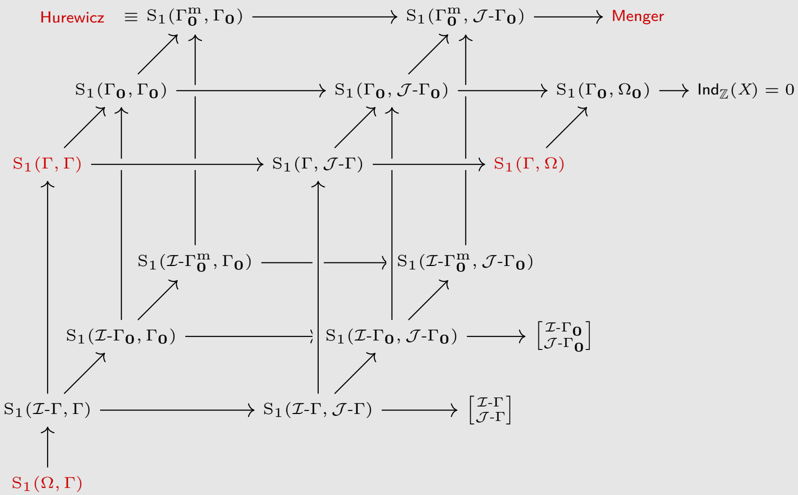

Diagram. The overall relations of investigated properties.

Cardinal invariants

- $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space) denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

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  - We say that φ goes through $\mathcal{I}$-slalom instead of φ Fin-goes through $\mathcal{I}$-slalom.

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$$\lambda(\mathcal{I}, \mathcal{J}) = \min \left\{ |\mathcal{R}| : \ \mathcal{R} \text{ contains } \mathcal{I}^d\text{-slaloms}, \ (\forall \varphi \in {}^{\omega}\omega)(\exists s \in \mathcal{R}) \ \neg(\varphi \ \mathcal{J}\text{-goes through } s) \right\}$$

- $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)\text{-space})$ denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

- a sequence $s \in {}^{\omega}\mathcal{A}$ will be called an $\mathcal{A}$-**slalom**.
- a function $\varphi \in {}^{\omega}\omega$ $\mathcal{J}$-**goes** through $\mathcal{A}$-slalom s if $\{n : \varphi(n) \in s(n)\} \in \mathcal{J}^d$, i.e., $\{n : \varphi(n) \in \omega \setminus s(n)\} \in \mathcal{J}$.
    - We say that φ goes through $\mathcal{I}$-slalom instead of φ Fin-goes through $\mathcal{I}$-slalom.

$$\mathfrak{b} = \min\left\{|\mathcal{R}| : \mathcal{R} \subseteq {}^{\omega}\omega,\ (\forall \mathrm{Fin}\text{-slalom } s)(\exists \varphi \in \mathcal{R})\ \neg(\varphi \text{ goes through } s)\right\}.$$

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$$\mathrm{cov}^*(\mathcal{I}) = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall S \in [\omega]^{\omega})(\exists A \in \mathcal{A})\ |S \cap A| = \omega\right\}.$$

Cardinal invariants

- $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)\text{-space})$ denotes the minimal cardinality of a perfectly normal space which is not an $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$-space.

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$$\mathfrak{k}_{\mathcal{I},\mathcal{J}} = \min\left\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \mathcal{A} \not\leq_K \mathcal{J}\right\}.$$

Cardinal invariants

- J. Šupina's results [12]: $\lambda(\mathrm{Fin}, \mathcal{J}) = \mathfrak{b}_{\mathcal{J}}$ and
  if $\mathcal{I}_1 \leq_K \mathcal{I}_2$ and $\mathcal{J}_1 \leq_{KB} \mathcal{J}_2$ then $\lambda(\mathcal{I}_2, \mathcal{J}_1) \leq \lambda(\mathcal{I}_1, \mathcal{J}_2)$.

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Theorem (V.Š., J.Šupina)

(1) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) \leq \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \mathfrak{b}_{\mathcal{J}}\}$.

(2) *If* $\mathcal{I} \not\leq_K \mathcal{J}$ *and* $\mathcal{J} \leq_K \mathcal{I}$ *then* $\lambda(\mathcal{I}, \mathcal{J}) = \min\{\mathfrak{k}_{\mathcal{I},\mathcal{J}}, \lambda(\mathcal{J}, \mathcal{J})\}$.

(3) *If* $\mathcal{I}$ *is tall then* $\lambda(\mathcal{I}, \mathrm{Fin}) = \min\{\mathtt{cov}^*(\mathcal{I}), \mathfrak{b}\}$.

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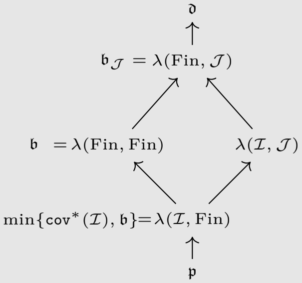

Diagram. Cardinal $\lambda(\mathcal{I}, \mathcal{J})$.

Critical cardinality

Theorem (V.Š., J.Šupina)

Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω, D being a discrete topological space. Then the following statements are equivalent.

(a) D *is an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$*-space.*

(b) $\mathrm{C}_p(D)$ *has* $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ s\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$.

(c) $\mathrm{C}_p(D)$ *has the property* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$

(d) $\mathrm{C}_p(D)$ *has the property* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$.

(e) $|D| < \lambda(\mathcal{I}, \mathcal{J})$.

Theorem (A. Kwela–M. Repický)

Let D be a discrete topological space. Then the following statements are equivalent.

(a) $|D| < \mathtt{cov}^*(\mathcal{I})$.

(b) $\mathrm{C}_p(D)$ *has* $\left[\begin{smallmatrix}\mathcal{I}\mathrm{QN}_{\mathbf{0}}\\ \mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]$.

(c) $\mathrm{C}_p(D)$ *has* $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \Gamma_{\mathbf{0}}\end{smallmatrix}\right]$.

(d) D *has the property* $\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma\\ \Gamma\end{smallmatrix}\right]$.

Critical cardinality

- Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals.
  (1) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\mathcal{I}\text{-}\Gamma_{\mathbf{0}}\\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \lambda(\mathcal{I}, \mathcal{J})$.
  (2) $\mathrm{non}(\mathrm{S}_1(\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[\begin{smallmatrix}\Gamma_{\mathbf{0}}\\ \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}\end{smallmatrix}\right]) = \mathfrak{b}_{\mathcal{J}}$.

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- If $\mathcal{I}$ is tall then
  (3) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{m}_{\mathbf{0}}, \Gamma_{\mathbf{0}})) = \mathrm{non}(\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right]) = \min\{\mathsf{cov}^*(\mathcal{I}), \mathfrak{b}\}$.
  (4) **(A. Kwela–M. Repický)** $\mathrm{non}(\left[{\mathcal{I}\mathrm{QN}_{\mathbf{0}} \atop \mathrm{QN}_{\mathbf{0}}}\right]) = \mathrm{non}(\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \Gamma_{\mathbf{0}}}\right]) = \mathrm{non}(\left[{\mathcal{I}\text{-}\Gamma \atop \Gamma}\right]) = \mathsf{cov}^*(\mathcal{I})$.

Critical cardinality

- Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$ be ideals.
  (1) $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma^{\mathrm{m}}_{\mathbf{0}}, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})) = \mathrm{non}(\left[{\mathcal{I}\text{-}\Gamma_{\mathbf{0}} \atop \mathrm{s}\mathcal{J}\mathrm{QN}_{\mathbf{0}}}\right]) = \lambda(\mathcal{I}, \mathcal{J})$.
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- Consistency
  (1) If $\mathfrak{b} = \mathfrak{c}$ then $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathsf{cov}^*(\mathcal{I})$ for every tall ideal $\mathcal{I}$.
  (2) If $\mathfrak{b} < \mathsf{cov}^*(\mathcal{I})$ then $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathsf{cov}^*(\mathcal{I})$ for every tall ideal $\mathcal{I}$.
  (3) If $\mathfrak{p} = \mathfrak{b}$ then $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) = \mathfrak{b}$.
  (4) If $\mathsf{cov}^*(\mathcal{I}) < \mathfrak{b}$ then $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \Gamma)) < \mathfrak{b}$.
  (5) If $\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$ then $\mathrm{non}(\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)) < \mathfrak{d}$.

Proposition (V.Š., J.Šupina)

(1) *If* $\mathfrak{p} < \mathfrak{b}$ *there is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*

(2) *If* $\mathtt{cov}^*(\mathcal{I}) < \mathfrak{b}$ *there is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*

(3) *For any* $\mathfrak{b}$*-Sierpiński set* S *there is an ultrafilter* $\mathcal{U}$ *such that* S *such that* $\mathrm{C}_p(S)$ *is not an* $\mathrm{S}_1(\mathcal{U}\text{-}\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$*-space (but* S *is an* $\mathrm{S}_1(\Gamma, \Gamma)$*-space).*

(4) *If* $\mathfrak{b} < \mathfrak{b}_{\mathcal{U}}$ *then there is an* $\mathrm{S}_1(\Gamma, \mathcal{U}\text{-}\Gamma)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*

(5) *If* $\mathfrak{b}_{\mathcal{J}} < \mathfrak{d}$ *then there is an* $\mathrm{S}_1(\Gamma, \Omega)$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\Gamma_{\mathbf{0}}^m, \mathcal{J}\text{-}\Gamma_{\mathbf{0}})$*-space.*

(6) *If* $\mathfrak{b} < \mathtt{cov}^*(\mathcal{I})$ *then there is an* $[\mathcal{I}\text{-}\Gamma, \Gamma]$*-space* X *such that* $\mathrm{C}_p(X)$ *is not an* $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma_{\mathbf{0}}^m, \Gamma_{\mathbf{0}})$*-space.*

Conclusion

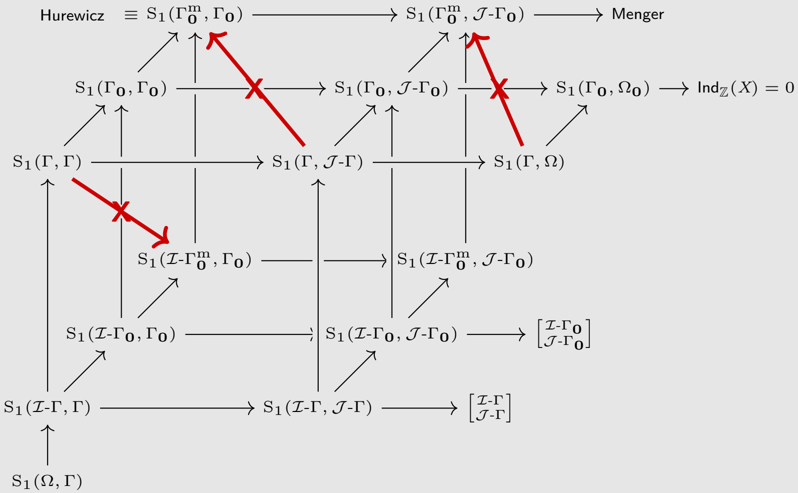

Diagram. The overall relations of investigated properties.

Thank you for your attention

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