Open Colorings, Perfect Sets and Games on Generalized Baire Spaces

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Generalized Baire spaces

Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$.

The **κ-Baire space** ${}^{\kappa}\kappa$ is the set of functions $f : \kappa \to \kappa$, with the **bounded topology:** basic open sets are of the form

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κ-**Borel sets:** close the family of open subsets under intersections and unions of size $\leq \kappa$ and complementation.

Open coloring axioms for subsets of the κ-Baire space

Let $X \subseteq {}^{\kappa}\kappa$.

$\mathrm{OCA}_\kappa(X)$:

Suppose $[X]^2 = R_0 \cup R_1$ is an open partition

(i.e. $\{(x, y) : \{x, y\} \in R_0\}$ is an open subset of $X \times X$).

Then either X is a union of κ-many R_1-homogeneous sets, or there exists an R_0-homogeneous set of size κ^+.

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i.e., there is a continuous embedding $f : {}^{\kappa}2 \to X$ whose image is R_0-homogeneous.

$\mathrm{OCA}^*_\kappa(X)$ for κ-analytic X

κ-**analytic** or $\mathbf{\Sigma}^1_1(\kappa)$ sets: continuous images of κ-Borel sets;
equivalently: continuous images of closed sets.

Theorem (Sz.)

If $\lambda > \kappa$ is inaccessible and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then

$$\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa)) \textit{ holds in } V[G].$$

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- In the classical setting (when $\kappa = \omega$), $\mathrm{OCA}^*(\mathbf{\Sigma}^1_1)$ holds in ZFC (Feng, 1993).
- For uncountable $\kappa = \kappa^{<\kappa}$, $\mathrm{OCA}^*_\kappa(\mathbf{\Sigma}^1_1(\kappa))$ is equiconsistent with the existence of an inaccessible $\lambda > \kappa$ by our result.

$\mathrm{OCA}^*_\kappa(X)$ for definable $X \subseteq {}^\kappa\kappa$

Work in progress

If $\lambda > \kappa$ is inaccessible and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then in $V[G]$,

$$\mathrm{OCA}^*_\kappa(X) \text{ holds for all } X \subseteq {}^\kappa\kappa \text{ definable from an element of } {}^\kappa\kappa.$$

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- The classical version of this result is due to Feng (1993).
- The κ-perfect set property holds for such subsets X (Schlicht, 2017).

Question

Let OCA_κ say: "$\mathrm{OCA}_\kappa(X)$ holds for all $X \subseteq {}^\kappa\kappa$".

Is OCA_κ consistent?

If so, how does it influence the structure of the κ-Baire space?

Perfectness for the κ-Baire space

A subset of ${}^{\kappa}\kappa$ is **closed** if and only if it is the set of branches

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A subtree T of ${}^{<\kappa}\kappa$ is a **strong κ-perfect tree** if T is $<\kappa$-closed and every node of T extends to a splitting node.

A set $X \subseteq {}^{\kappa}\kappa$ is a **strong κ-perfect set** if $X = [T]$ for a strong κ-perfect tree T.

Väänänen's perfect set game

Perfectness was first generalized for the κ-Baire space by Väänänen, based on the following game.

Definition (Väänänen)

Let $X \subseteq {}^{\kappa}\kappa$, let $x_0 \in X$ and let $\omega \leq \gamma \leq \kappa$. Then $\mathcal{V}_\gamma(X, x_0)$ is the following game.

$$\begin{array}{lcccccccccc} \mathbf{I} & & U_1 & & \ldots & & U_\alpha & & \ldots & \\ \mathbf{II} & x_0 & & x_1 & & \ldots & & x_\alpha & & \ldots \end{array}$$

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$\mathbf{I}$		U_1		$\dots$		U_α		$\dots$
$\mathbf{II}$	x_0		x_1		$\dots$		x_α		$\dots$

$\mathbf{I}$ plays a basic open sets U_α of X such that $U_\alpha \subsetneq U_\beta$ for all $\beta < \alpha$, and $x_\beta \in U_{\beta+1}$ at successor rounds $\alpha = \beta + 1$, and $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$ at limit rounds α.

$\mathbf{II}$ responds with $x_\alpha \in U_\alpha$ such that $x_\alpha \neq x_\beta$ for all $\beta < \alpha$.

Player $\mathbf{II}$ wins the run if she can make all her γ moves legally.

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Let $X \subseteq {}^{\kappa}\kappa$ and let $\omega \leq \gamma \leq \kappa$.

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X is **γ-perfect** if $\mathbf{II}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

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X is γ-**perfect** if $\mathbf{II}$ wins $\mathcal{V}_{\gamma}(X, x_0)$ for all $x_0 \in X$.

X is γ-**scattered** if $\mathbf{I}$ wins $\mathcal{V}_{\gamma}(X, x_0)$ for all $x_0 \in X$.

Definition

Let T be a subtree of ${}^{<\kappa}\kappa$, and let $t_0 \in T$. Then $\mathcal{G}^*_\kappa(T, t_0)$ is the following game.

$$
\begin{array}{lccccccc}
\mathbf{I} & & i_1 & & \dots & & i_\alpha & & \dots \\
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$\mathbf{II}$ plays $t_\alpha^0, t_\alpha^1 \in T$ such that $t_\alpha^0 \perp t_\alpha^1$ and $t_\alpha^i \supset t_\beta^{i_\beta}$ for all $\beta < \alpha$ and $i < 2$. Then $\mathbf{I}$ chooses, by playing $i_\alpha < 2$.

(Thus, $\mathbf{II}$ plays a pair of disjoint basic open subsets of $[T]$ which are contained in the previously chosen basic open sets).

κ-perfect and κ-scattered trees

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κ-perfect sets and trees

Proposition

Let $X \subseteq {}^{\kappa}\kappa$. The following are equivalent.

1. *X is a κ-perfect set.*
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This may not hold for κ-scattered sets and trees.

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Thus, the two games lead to equivalent definitions of κ-perfectness and κ-scatteredness for trees.

γ-perfect sets and trees when $\gamma < \kappa$

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Let $X \subseteq {}^\kappa\kappa$ and let $\omega \leq \gamma < \kappa$.

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2. *If κ is weakly compact and $X \subseteq {}^{<\kappa}2$, then*

$$X \text{ is a } \gamma\text{-perfect set} \iff X = [T] \text{ for a } \gamma\text{-perfect tree } T.$$

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More generally: if κ has the tree property and T is a κ-tree, then

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3. *Analogue of these statements for "generalized Cantor-Bendixson ranks" for subsets of ${}^\kappa\kappa$ and for subtrees of ${}^{<\kappa}\kappa$.*

Generalized Cantor-Bendixson hierarchies can be defined for subsets of the κ-Baire space and for subtrees of ${}^{<\kappa}\kappa$, using modifications of Väänänen's and Galgon's games.

Väänänen's generalized Cantor-Bendixson theorem

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The following statements are equivalent:

1. *The κ-perfect set property for closed subsets of ${}^{\kappa}\kappa$ (every closed subset of ${}^{\kappa}\kappa$ of size $> \kappa$ has a κ-perfect subset).*

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- Galgon (2016) showed that (2) holds after Lévy-collapsing an inaccessible $\lambda > \kappa$ to κ^+.

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Remark: The following are equivalent for any $X \subseteq {}^{\kappa}\kappa$.

- X contains a κ-dense in itself subset.
- X contains a subset whose closure is a strong κ-perfect set.
- Player $\mathbf{II}$ wins Väänänen's game $\mathcal{V}_{\kappa}(X, x)$ for some $x \in X$.

Thank you!