A fragment of Asperó-Mota's Finitely Proper Forcing Axiom and an entangled set of reals

Teruyuki Yorioka (Shizuoka U.)

FACULTY OF SCIENCE
SHIZUOKA UNIVERSITY

Novi Sad Conference in Set Theory and General Topology

University of Novi Sad

July 2nd 2018, 18:30 – 18:55

Motivation: Two topics on uncountable linear orders

Theorem

The assertion (B) *means that any two* $\aleph_1$*-dense sets of reals are order-isomorphic.*

(Baumgartner) (B) *is consistent.*

(Todorčević) PFA *implies* (B).

(Abraham-Shelah) *It is consistent that* $\mathrm{MA}_{\aleph_1}$ *holds and* (B) *fails.*

Theorem

The assertion (T) *means that every Countryman line contains an isomorphic copy of Todorčević's Countryman line* $C(\rho_0)$ *or its reverse.*

(Todorčević) PFA *implies* (T)*, in particular,* $\mathrm{MA}_{\aleph_1}$ *combining with the assertion that any two Aronszajn trees are club-isomorphic implies* (T).

(Peng) *It is consistent that* $\mathrm{MA}_{\aleph_1}$ *holds and* (T) *fails.*

Motivation: Two topics on uncountable linear orders

Theorem (Asperó-Mota)

Define $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ *which satisfies that*

- $\mathrm{PFA} \Rightarrow \mathrm{PFA}^{\mathrm{fin}}(\aleph_1) \Rightarrow \mathrm{MA}_{\aleph_1}$, *and converse implications may fail,*
- *it is consitent that* $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ *(in particular* $\mho$*) holds and* $2^{\aleph_0} > \aleph_2$.

(Todorčević) PFA implies (B).

Question. Does $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ imply (B)?

(Abraham-Shelah) It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and (B) fails.

(Todorčević) PFA implies (T).

Proposition. $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ implies (T).

(Peng) It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and (T) fails.

(B) Any two $\aleph_1$-dense sets of reals are order-isomorphic.

(T) Every Countryman line contains an isomorphic copy of Todorčević's Countryman line $C(\rho_0)$ or its reverse.

Definition (Abraham-Shelah)

Let $k \in \mathbb{N}$, $k \geq 2$. An uncountable set E of reals is called *k-entangled* if, for any pairwise disjoint $\{\sigma_\xi : \xi \in \omega_1\} \subseteq [E]^k$ and any $d \in {}^k\{0,1\}$, there is $\{\xi, \eta\} \in [\omega_1]^2$ such that $\sigma_\xi \not\leq_d \sigma_\eta$: Either $\forall i < k\Big(d(i) = 0 \leftrightarrow$ (i-th element of σ_ξ) $<$ (i-th element of σ_η)$\Big)$ or $\forall i < k\Big(d(i) = 0 \leftrightarrow$ (i-th element of σ_η) $<$ (i-th element of σ_ξ)$\Big)$.

Proposition

A k-entangled set of reals gives a counterexample of (B).

Lemma (Abraham-Shelah)

For a ccc forcing notion $\mathbb{P}$, if $\mathbb{P}$ destroys a k-entangledness of E, then there exists a ccc forcing notion $\mathcal{A}(\mathbb{P})$ that adds an uncountable antichain of $\mathbb{P}$ and forces that E is still k-entangled.

Theorem (Abraham-Shelah)

For each integer $k \geq 2$, it is consistent that $\mathrm{MA}_{\aleph_1}$ holds and there exists a k-entangled set of reals.

Lemma

For a forcing notion $\mathbb{P}$, if $\mathbb{P}$ destroys a k-entangledness of E, then there exists a proper forcing notion $\mathcal{A}(\mathbb{P})$ which forces that $\mathbb{P}$ collapses ω_1 and E is still k-entangled.

Proof. Let P be a forcing notion such that $\not\Vdash_P$ " E is k-entangled ". Then there are $p \in P$, $d \in {}^k\{0,1\}$ and a P-name $\dot{I}$ such that

$$p \Vdash_P \text{" } \dot{I} \subseteq [\omega_1]^k \text{ is a pairwise disjoint uncountable } \perp_d\text{-homogeneous set ".}$$

Define

$$S(P,p,d,\dot{I}) = S(P) := \left\{ \langle q, \Sigma, n \rangle \in P \times \left[[\omega_1]^k \right]^{k+1} \times \omega : q \le_P p \;\&\; q \Vdash_P \text{" } \Sigma \subseteq \dot{I} \text{"} \right\}.$$

- $p \Vdash_P$ " $\dot{I} \subseteq [\omega_1]^k$ is a pairwise disjoint uncountable $\perp_d$-homogeneous set ",
- $S(P, p, d, \dot{I}) = S(P) := \left\{ \langle q, \Sigma, n \rangle \in P \times \left[[\omega_1]^k \right]^{k+1} \times \omega : q \leq_P p \; \& \; q \Vdash_P \text{“} \Sigma \subseteq \dot{I} \text{”} \right\}$.

$\mathcal{A}(P, p, d, \dot{I}) = \mathcal{A}(P)$ that consists of the pairs $\langle \mathcal{N}, W \rangle$ such that

- $\mathcal{N}$ is a finite $\in$-chain of countable elementary submodels of H_λ which contains the set $\left\{ E, P, p, d, \dot{I} \right\}$,
- $W \in [S(P)]^{<\aleph_0}$, and, for each $x \in W$, write $x = \langle q^x, \Sigma^x, n^x \rangle$,
- for each $x \in W$, Σ^x is separated by $\mathcal{N}$,
- the set $\{\Sigma^x : x \in W\}$ is also separated by $\mathcal{N}$, and
- for any $\{x, y\} \in [W]^2$, if $\Sigma^x \cup \Sigma^y$ is $\perp_d$-homogeneous, then $n^x \neq n^y$,

$$\langle \mathcal{N}, W \rangle \leq_{\mathcal{A}(P,p,d,\dot{I})} \langle \mathcal{N}', W' \rangle :\Longleftrightarrow \mathcal{N} \supseteq \mathcal{N}' \; \& \; W \supseteq W'.$$

Note that $\mathcal{A}(P)$ is proper and preserves E to be k-entangled.

Review 1: (Abraham-Shelah). Con($MA_{\aleph_1}$ & $\neg$(B)).

- $p \Vdash_P$ " $\dot{I} \subseteq [\omega_1]^k$ is a pairwise disjoint uncountable $\perp_d$-homogeneous set",
- $S(P, p, d, \dot{I}) := \left\{ \langle q, \Sigma, n \rangle \in P \times \left[[\omega_1]^k \right]^{k+1} \times \omega : q \leq_P p \ \& \ q \Vdash_P \text{“} \Sigma \subseteq \dot{I} \text{”} \right\}$,
- For each $\langle \mathcal{N}, W \rangle \in \mathcal{A}(P, p, d, \dot{I}) = \mathcal{A}(P)$ and $x = \langle q^x, \Sigma^x, n^x \rangle$, $y = \langle q^y, \Sigma^y, n^y \rangle$ in W, if $\Sigma^x \cup \Sigma^y$ is $\perp_d$-homogeneous, then $n^x \neq n^y$.

For $x, y \in S(P, p, d, \dot{I})$, if $q^x \not\perp_P q^y$, then $n^x \neq n^y$ holds, because for $r \leq_p q^x, q^y$,

$$r \Vdash_P \text{“} \Sigma^x \cup \Sigma^y \subseteq \dot{I}, \text{ hence } \Sigma^x \cup \Sigma^y \text{ is } \perp_d\text{-homogeneous”}.$$

Therefore,

$$\Vdash_{\mathcal{A}(P)} \text{“} \Vdash_P \text{“ let } \dot{Y} := \left\{ \Sigma : \langle q, \Sigma, n \rangle \in \bigcup \dot{G}_{\mathcal{A}(P)} \text{ with } q \in \dot{G}_P \right\},$$

$$\text{then } \left\{ \max(\bigcup \Sigma) : \Sigma \in \dot{Y} \right\} \text{ is cofinal in } {\omega_1}^V, \textit{ and}$$

$$\begin{array}{ccll} \dot{Y} & \to & \omega & \text{is injective ” ”.} \\ \Downarrow & & \Downarrow & \\ \Sigma & \mapsto & n, \text{ which is the unique } n & \\ & & \text{so that } \langle q, \Sigma, n \rangle \in \bigcup \dot{G}_{\mathcal{A}(P)} \text{ with } q \in \dot{G}_P & \end{array}$$

□

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$).

Definition (Asperó-Mota)

- A forcing notion $\mathbb{P}$ is called *finitely proper* if, for any large enough regular cardinal λ, any finite set $\{N_i : i \in n\}$ of countable elementary submodels of H_λ containing $\mathbb{P}$ as a member, and any $p \in \mathbb{P} \cap \bigcap_{i \in n} N_i$, there exists an extension of p in $\mathbb{P}$ that is $(N_i, \mathbb{P})$-generic for all $i \in n$.
- $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ is the forcing axiom for all finitely proper forcing notions of size $\aleph_1$ and $\aleph_1$-many dense sets.

Proposition

The following assertions follow from $\mathrm{PFA}^{fin}(\aleph_1)$*.*

- $\mathrm{MA}_{\aleph_1}$,
- $\mho$,
- *there are no weak club guessing ladder systems,*
- *any two Aronszajn trees are club isomorphic.*

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$).

To force $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ together with $2^{\aleph_0} > \aleph_2$, use an iteration of V-finitely proper forcing notions *by finite support equipped with models as side conditions*.

Suppose CH. Let

- κ be an uncountable regular cardinal with $\kappa \geq \aleph_2$ and $2^{<\kappa} = \kappa$,
- $\Phi : \kappa \to H_\kappa$ be a surjection such that, for any $x \in H_\kappa$, $\Phi^{-1}[\{x\}]$ is unbounded in κ,
- $\mathcal{M}_0 := \{M \in [H_\kappa]^{\aleph_0} : M \prec (H_\kappa, \Phi)\}$.

Definition (Todorčević, Asperó-Mota)

A finite subset S of $\mathcal{M}_0$ is called a *symmetric system* if

- for each $M, M' \in S$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $M \simeq M'$,
- for each $M, M' \in S$, if $\omega_1 \cap M' < \omega_1 \cap M$, then there exists $M'' \in S$ such that $M'' \simeq M$ and $M' \in M''$,
- for each $M_0, M_1 \in S$ and $M' \in S \cap M_0$, if $\omega_1 \cap M_0 = \omega_1 \cap M_1$, then $\Psi_{M_0,M_1}(M') \in S$,
- for each $M, M' \in S$, if $\omega_1 \cap M = \omega_1 \cap M'$, then $\Psi_{M,M'} \restriction (\kappa \cap M \cap M')$ is identity.

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$).

By induction on $\alpha \in \kappa$, define $\mathbb{P}_\alpha$ which consists of $p = (R_p, A_p)$ such that

1. $R_p \subseteq \mathcal{M}_0 \times \alpha$, $\mathrm{dom}(R_p)$ is a symmetric system and, for each $M \in \mathrm{dom}(R_p)$, the range of $R \cap (\{M\} \times \alpha)$ is an initial segment of $\alpha \cap M$,
2. A_p is a function with domain a finite subset of α such that, for any $\xi \in \mathrm{dom}(A_p)$, $\Phi(\xi)$ is a $\mathbb{P}_\xi$-name for a V-finitely proper forcing notion on ω_1, and if $p \restriction \xi \in \mathbb{P}_\xi$, then for any $M \in {R_p}^{-1}[\{\xi\}] := \{M \in \mathrm{dom}(R_p) : \langle M, \xi \rangle \in R_p\}$,
$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“ } A_p(\xi) \text{ is } (M[\dot{G}], \Phi(\xi))\text{-generic ”},$$

$$q \leq_{\mathbb{P}_\alpha} p \iff R_q \supseteq R_p \text{ \& for any } \xi \in \mathrm{dom}(A_p),$$
$$q \restriction \xi \leq_{\mathbb{P}_\xi} p \restriction \xi \text{ \& } q \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“ } A_q(\xi) \leq_{\Phi(\xi)} A_p(\xi) \text{ ”}.$$

Note that $\mathbb{P}_\alpha$ has $\left(2^{\aleph_0}\right)^+$-cc, hence CH implies that $\mathbb{P}_\alpha$ has $\aleph_2$-cc.

Want to show that, for any $p \in \mathbb{P}_\alpha$, $\xi \in \mathrm{dom}(A_p)$, and $M \in {R_p}^{-1}[\{\xi\}]$, p is $(M, \mathbb{P}_\xi)$-generic. However, if M doesn't have enough information on $\mathbb{P}_\xi$, it would not be possible.

- $\theta_0 = (2^\kappa)^+$ and $\theta_\alpha := \left(2^{\sup\{\theta_\beta ; \beta \in \alpha\}}\right)^+$ for each $\alpha \in \kappa$,
- for each $\alpha \in \kappa$, $\mathcal{M}^*_\alpha := \left\{N^* \in [H_{\theta_\alpha}]^{\aleph_0} : N^* \prec H_{\theta_\alpha}, \{\Phi, \langle \theta_\xi : \xi < \alpha \rangle\} \in N^*\right\}$, $\mathcal{M}_\alpha := \{N^* \cap H_\kappa : N^* \in \mathcal{M}^*_\alpha\}$.

Definition (Asperó-Mota)

By induction on $\alpha \in \kappa$, define $\mathbb{P}_\alpha$ which consists of $p = (R_p, A_p)$ such that

1. $R_p \subseteq \mathcal{M}_0 \times \alpha$, $\mathrm{dom}(R_p)$ is a symmetric system and, for each $M \in \mathrm{dom}(R_p)$, the range of $R \cap (\{M\} \times \alpha)$ is an initial segment of $\alpha \cap M$ such that, for any $\xi < \alpha$, ${R_p}^{-1}[\{\xi\}] := \{M \in \mathrm{dom}(R_p) : \langle M, \xi \rangle \in R_p\} \subseteq \mathcal{M}_\xi$,
2. A_p is a function with domain a finite subset of α such that, for any $\xi \in \mathrm{dom}(A_p)$, $\Phi(\xi)$ is a $\mathbb{P}_\xi$-name for a V-finitely proper forcing notion on ω_1, and if $p \restriction \xi \in \mathbb{P}_\xi$, then for any $M \in {R_p}^{-1}[\{\xi\}]$,

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“} A_p(\xi) \text{ is } (M[\dot{G}], \Phi(\xi))\text{-generic”},$$

$$q \leq_{\mathbb{P}_\alpha} p \iff R_q \supseteq R_p \ \&\ \text{for any } \xi \in \mathrm{dom}(A_p),$$
$$q \restriction \xi \leq_{\mathbb{P}_\xi} p \restriction \xi \ \&\ q \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“} A_q(\xi) \leq_{\Phi(\xi)} A_p(\xi) \text{”}.$$

Theorem (Asperó-Mota)

1. *For $\alpha < \beta < \kappa$, $\mathbb{P}_\alpha$ completely embeds into $\mathbb{P}_\beta$.*
2. *For $N^* \in \mathcal{M}^*_{\alpha+1}$ and $p \in \mathbb{P}_\alpha$,*
   if $\{N^* \cap H_\kappa\} \times (\alpha \cap N^*) \subseteq R_p$, then p is $(N^*, \mathbb{P}_\alpha)$-generic.
   In particular, if $\langle N^* \cap H_\kappa, \alpha\rangle \in R_p$, p is $(N^*, \mathbb{P}_\alpha)$-generic.
3. *Let $\mathbb{P}^*_\kappa$ be the direct limit of the $\mathbb{P}_\alpha$. Then $\mathbb{P}^*_\kappa$ is proper and forces that $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ holds and $2^{\aleph_0} = \kappa$.*

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$), and Result

Let E be a k-entangled set of reals. Want to force $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ combining with preserving E to be k-entangled.

For $\alpha < \kappa$, define Asperó-Mota iteration $\mathbb{P}^E_\alpha$ such that, at stage $\xi < \alpha$,

if $\Phi(\xi)$ is a $\mathbb{P}^E_\xi$-name for a V-finitely proper forcing notion on ω_1 and preserves E to be k-entangled, then force $\Phi(\xi)$,

if $\Phi(\xi)$ is a $\mathbb{P}^E_\xi$-name for a V-finitely proper forcing notion on ω_1 and destorys the k-entangledness of E, then force $\mathcal{A}(\Phi(\xi))$ whose conditions $(\mathcal{N}, W)$ satisfy that $\mathcal{N} \subseteq {R_p}^{-1}[\{\xi\}]$. (This seems to be necessary to show that $\mathbb{P}^E_\alpha$ preserves E to be k-entangled.)

Attention 1.

For $p = (R_p, A_p) \in \mathbb{P}_\alpha$ and $M \in \mathrm{dom}(R_p)$, the marker of M cannot be increased freely. Because a condition $p \in \mathbb{P}_\alpha$ is required that, for any $M \in {R_p}^{-1}[\{\xi\}]$,

$$p \restriction \xi \Vdash_{\mathbb{P}_\xi} \text{“ } A_p(\xi) \text{ is } (M[\dot{G}], \Phi(\xi))\text{-generic ”}.$$

So to accomplish $* * *$ above, a condition $p = (R_p, A_p)$ of $\mathbb{P}^E_\alpha$ should be required that, for each $\xi < \alpha$, ${R_p}^{-1}[\{\xi\}]$ forms a symmetric system.

Attention 2.

Even if $M, N_0, N_1 \in \mathcal{M}_\xi$, $M \in N_0$ and $N_0 \simeq N_1$, $\Psi_{N_0,N_1}(M)$ may NOT be in $\mathcal{M}_\xi$.

Hence we introduce a new $\mathcal{M}_\xi$ and give up forcing the whole $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$.

Definition (Miyamoto)

Define a symmetric system of countable elementary submodels of the relational structure like

$$\left\langle H_\kappa, \in, \mathbb{P}, \leq_{\mathbb{P}}, R^{\mathbb{P}}_{=}, R^{\mathbb{P}}_{\in}, H^{\mathbb{P}}_{\kappa}, E, \Phi \right\rangle,$$

and define $\mathcal{M}^P_\alpha$, for each $\alpha < \kappa$, with the property:

- For any $M, N_0, N_1 \in \mathcal{M}^P_\xi$ with $M \in N_0$ and $N_0 \simeq N_1$, $\Psi_{N_0,N_1}(M)$ is in $\mathcal{M}^P_\xi$,
- Let $\mathbb{P}_\alpha$ be Asperó-Mota iteration with the property that, for each $\xi < \alpha$, ${R_p}^{-1}[\{\xi\}]$ forms a symmetric system.

  Under some assumption, it is possible to show that $\mathbb{P}_\alpha$ is proper as a class forcing, more precisely, for any $N \in \mathcal{M}^P_\alpha$ and $p \in \mathbb{P}_\alpha$, if $\{N\} \times (\alpha \cap N) \subseteq R_p$, then p is $(N, \mathbb{P}_\alpha)$-generic.

Definition

Define $\mathbb{P}^E_\alpha$ as above, but conditions $p = (R_p, A_p)$ satisfies that for each $\xi < \alpha$, ${R_p}^{-1}[\{\xi\}]$ forms a symmetric system.

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$), and Result

Definition

1. For a forcing notion $\mathbb{P}$, a countable elementary submodel M of H_λ containing $\mathbb{P}$ as a member, and $p \in \mathbb{P}$, p is called a *solid* $(M, \mathbb{P})$-generic condition if, for any countable elementary submodel N of H_λ containing $\mathbb{P}$ as a member with $\omega_1 \cap N = \omega_1 \cap M$, p is $(N, \mathbb{P})$-generic.
2. A forcing notion $\mathbb{P}$ is *s-finitely proper* if, for every large enough regular cardinal λ, every finite set $\{N_i : i \in n\}$ of countable elementary submodels of H_λ containing $\mathbb{P}$ as a member, and every $p \in \mathbb{P} \cap \bigcap_{i \in n} N_i$, there exists an extension of p in $\mathbb{P}$ that is *solid* $(N_i, \mathbb{P})$-generic for all $i \in n$.
3. $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ is the forcing axiom for all s-finitely proper forcing notions of size $\aleph_1$ and $\aleph_1$-many dense sets.

Proposition

$\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ also *implies* $\mathrm{MA}_{\aleph_1}$, $\mho$, *that there are no weak club guessing ladder systems, and that any two Aronszajn trees are club isomorphic.*

It is not known for sure whether $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ is equivalent to $\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$.

Definition

Define $\mathbb{P}^E_\alpha$ as above, but replace V-finitely proper with V-s-finitely proper in the definition.

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$), and Result

Theorem (Miyamoto-Y.)

1. *For $\alpha < \beta < \kappa$, $\mathbb{P}^E_\alpha$ completely embeds into $\mathbb{P}^E_\beta$.*
2. For $N \in \mathcal{M}^P_{\alpha+1}$ and $p \in \mathbb{P}^E_\alpha$, if $\{N \cap H_\kappa\} \times (\alpha \cap N) \subseteq R_p$, then is p $(N, \mathbb{P}^E_\alpha)$-generic? To show this, a trouble would be happened when α is $\geq \omega_2$ and is of uncountable cofinality.
   For a predense subset $D \in N$ of $\mathbb{P}^E_\alpha$, we have to build an extension q, that is compatible with some condition in D, of a given p such that, for any $\xi < \alpha$, ${R_q}^{-1}[\{\xi\}]$ forms a symmetric system.

Theorem (Miyamoto-Y.)

1. *For $\alpha < \beta < \kappa$, $\mathbb{P}^E_\alpha$ completely embeds into $\mathbb{P}^E_\beta$.*
2. *If $\alpha < \omega_2$, then for $N \in \mathcal{M}^P_{\alpha+1}$ and $p \in \mathbb{P}^E_\alpha$, if $\{N \cap H_\kappa\} \times (\alpha \cap N) \subseteq R_p$, then p is $(N, \mathbb{P}^E_\alpha)$-generic.*
3. *Suppose that $\kappa = \aleph_2$, and Let $\mathbb{P}^{E*}_\kappa$ be the direct limit of the $\mathbb{P}^E_\alpha$. Then $\mathbb{P}^{E*}_\kappa$ is proper, preserves E to be k-entangled, and forces that $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ holds and $2^{\aleph_0} = \kappa = \aleph_2$. Consequently, for each integer $k \geq 2$, it is consistent that $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ holds, $2^{\aleph_0} = \kappa = \aleph_2$, and there exists a k-entangled set of reals.*

Review 2: (Asperó-Mota). Con($\mathrm{PFA}^{\mathrm{fin}}(\aleph_1)$ & $2^{\aleph_0} > \aleph_2$), and Result

(Todorčević) PFA implies (B).

(Miyamoto-Y.) $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ imply (B).

(Abraham-Shelah) It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and (B) fails.

(Todorčević) PFA implies (T).

Proposition. $\mathrm{PFA}^{\text{s-fin}}(\aleph_1)$ implies (T).

(Peng) It is consistent that $\mathrm{MA}_{\aleph_1}$ holds and (T) fails.

(B) Any two $\aleph_1$-dense sets of reals are order-isomorphic.

(T) Every Countryman line contains an isomorphic copy of Todorčević's Countryman line $C(\rho_0)$ or its reverse.