Some results on the Baire Rado's Conjecture

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Introduction

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1. is *non-trivial* if each $t \in T$ has two incompatible extensions;
2. does not split on the limit levels if for each limit α and $s, s' \in T$ such that $ht_T(s) = ht_T(s') = \alpha$, if $\{t \in T : t < s\} = \{t \in T : t < s'\}$, then $s = s'$.

In this talk, we will focus on trees of height ω_1 that are non-trivial and do not split on the limit levels.

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Remark

Note that a tree is Baire iff it is countably distributive as a forcing notion, i.e. it does not add any new countable sequence of ordinals.

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$RC \rightarrow RC^b$.

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The strength and limitations of RC^b:

Theorem

RC^b implies:

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3. *the Singular Cardinal Hypothesis (Todorcevic).*
4. *$\square(\lambda)$ fails for all regular $\lambda \geq \omega_2$ (Todorcevic) and in fact $\neg\square(\lambda, \omega)$ (Torres-Perez and Wu) and along with $\neg CH$, $\neg\square(\lambda, \omega_1)$ (Weiss) but not $\neg\square(\lambda, \omega_2)$ (Folklore).*

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7. *NS_{ω_1} is presaturated (Feng).*
8. $\binom{\omega_2}{\omega_1} \to \binom{\omega}{\omega}^{1,1}_{\omega}$ *and* $\binom{\omega_2}{\omega_1} \to \binom{k}{\omega_1}^{1,1}_{\omega}$ *for any* $k \in \omega$*, namely* $\forall f : \omega_2 \times \omega_1 \to \omega$*, there exist* $A \in [\omega_2]^{\omega}, B \in [\omega_1]^{\omega}$ *such that* $f \restriction A \times B$ *is constant (Todorcevic from CC, or Z. from the existence of a presaturated ideal) but not*

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8 $\binom{\omega_2}{\omega_1} \rightarrow \binom{\omega}{\omega}^{1,1}_{\omega}$ *and* $\binom{\omega_2}{\omega_1} \rightarrow \binom{k}{\omega_1}^{1,1}_{\omega}$ *for any* $k \in \omega$*, namely* $\forall f : \omega_2 \times \omega_1 \rightarrow \omega$*, there exist* $A \in [\omega_2]^{\omega}, B \in [\omega_1]^{\omega}$ *such that* $f \restriction A \times B$ *is constant (Todorcevic from CC, or Z. from the existence of a presaturated ideal) but not* $\binom{\omega_2}{\omega_1} \rightarrow \begin{bmatrix}\omega\\ \omega_1\end{bmatrix}^{1,1}_{\omega_1}$*, aka for all* $f : \omega_2 \times \omega_1 \rightarrow \omega_1$ *there exist* $A \in [\omega_2]^{\omega}$ *and* $B \in [\omega_1]^{\omega_1}$ *such that* $f'' A \times B \neq \omega_1$ *(Z.).*

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Torres-Pérez asked: How much fragment of *MA* is compatible with *RC*?

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10. and more ...

Torres-Pérez asked: How much fragment of *MA* is compatible with *RC*?
We are motivated by the second question with *RC* replaced by RC^b and *MA* replaced by *PFA*.

Known models of RC^b

RC^b is known to be consistent with CH and $\neg CH$. The following (due to Todorcevic) are models of RC^b (in fact RC):

1. $Coll(\omega_1, < \kappa)$ where κ is a strongly compact cardinal.
2. $\mathbb{M}(\omega_1, < \kappa)$ where κ is a strongly compact cardinal and the forcing is the Mitchell forcing (mixed support iteration) to get the tree property at ω_2.

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To show RC^b holds in these models, it is crucial to prove appropriate versions of "Baire preservation theorems".

Baire preservation lemma

Definition

A poset $\mathbb{P}$ is *countably capturing* if for any $p \in \mathbb{P}$, any $\mathbb{P}$-name of a countable sequence of ordinals $\dot{\tau}$, there exists another $\mathbb{P}$-name $\dot{\sigma}$ such that $|\dot{\sigma}| \leq \aleph_0$, and $q \leq p$ such that $q \Vdash_{\mathbb{P}} \dot{\tau} = \dot{\sigma}$.

Remark

Here we think of each $\mathbb{P}$-name $\dot{\tau}$ for a countable sequence of ordinals as represented by a function $f_{\dot{\tau}}$ whose domain is ω such that for each $n \in \omega$, $f_{\dot{\tau}}(n) = \{(\alpha_p, p) : p \in A_n\}$ where A_n is some antichain chain of $\mathbb{P}$ such that for each $p \in A_n$, $p \Vdash_{\mathbb{P}} \dot{\tau} = \alpha_p$. By saying $|\dot{\sigma}| \leq \aleph_0$, we really mean $|f_{\dot{\sigma}}| \leq \aleph_0$.

Remark

Any proper forcing is countably capturing.

Baire preservation lemma

Lemma

Let $\mathbb{P}$ be countably capturing and $\mathbb{Q}$ be countably distributive. Then TFAE:

1. $\Vdash_{\mathbb{P}} \check{\mathbb{Q}}$ *is countably distributive*
2. $\Vdash_{\mathbb{Q}} \check{\mathbb{P}}$ *is countably capturing.*

Sketch of one direction.

2) implies 1): Let $G \times H$ be generic for $\mathbb{P} \times \mathbb{Q}$ and let $\dot{\tau}$ be a $(\mathbb{P} \times \mathbb{Q})$-name of a countable sequence of ordinals. We need to show $(\dot{\tau})^{G \times H}$ is in $V[G]$. Since $\Vdash_{\mathbb{Q}} \mathbb{P}$ is countably capturing, in $V[H]$ (view $(\dot{\tau})^H$ as a $\mathbb{P}$-name), there exists a nice $\mathbb{P}$-name $\dot{\sigma}$ with $|\dot{\sigma}| \leq \aleph_0$ such that in $V[H][G]$, $(\dot{\tau})^{H \times G} = (\dot{\sigma})^G$. Since Q is countably distributive, $\dot{\sigma} \in V$. But then $(\dot{\tau})^{H \times G} = (\dot{\sigma})^G \in V[G]$.

□

First try: Separate RC^b from RC

Definition

Let $\sigma\mathbb{R}$ denote the tree consisting of bounded subsets of $\mathbb{R}$ well ordered by the natural order on $\mathbb{R}$. The tree is ordered by end-extension.

Observation

1. $\sigma\mathbb{R}$ is nonspecial (Kurepa);
2. $\sigma\mathbb{R}$ is not Baire;

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1. $\sigma\mathbb{R}$ is nonspecial (Kurepa);
2. $\sigma\mathbb{R}$ is not Baire;

Given a tree T, let $S(T)$ denote the Baumgartner specializing poset of T. More precisely, it contains finite functions $s : T \to \omega$ that are injective on chains.

Theorem (Baumgartner)

$S(T)$ is c.c.c iff T does not contain an uncountable branch.

First try: Separate RC^b from RC

Let κ be a strongly compact cardinal. Let $\langle P_i, \dot{Q}_j : i \leq \kappa, j < \kappa \rangle$ be finite support iteration of c.c.c forcing of length κ such that $\Vdash_{P_i} \dot{Q}_i = S(\sigma\mathbb{R})$.

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In $V^{\mathbb{P}_\kappa}$, all $< \kappa$-sized subset of $\sigma\mathbb{R}$ is special and any Baire tree T, there exists a nonspecial subtree of size $< \kappa$.
But we need to collapse κ to $\aleph_2$! No problem! We can do a mixed support iteration in the style of Mitchell.

Corollary (Z.)

RC^b does not imply RC.

Enlarge the fragment

The model presented above is not satisfactory: it only contains a small fragment of MA. There are a lot more forcings that preserve Baire trees that are not included.

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Recall for a Suslin tree S, the Suslinity of S is preserved under CS-iteration.

Enlarge the fragment

The model presented above is not satisfactory: it only contains a small fragment of MA. There are a lot more forcings that preserve Baire trees that are not included.
Recall for a Suslin tree S, the Suslinity of S is preserved under CS-iteration.
Ambitious: For a fixed Baire tree T, what if we try to iterate proper forcings that preserve the Baireness of T? Is the property preserved under CS-iteration?

No. :-(

For any Aronszajn tree T and any stationary subset $S \subset \omega_1$, the S-specializing poset $Q(T, S)$, due to Shelah, is a proper forcing that adds a regressive function on S, namely in $V^{Q(T,S)}$, there exists $S_1 \subset S$ such that $S - S_1$ is nonstationary and a function f defined on $T \restriction S_1$ such that $f(t) < ht_T(t)$ and any $t <_T t' \in dom(f)$, $f(t) \neq f(t')$.

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Example

Let T be a Suslin tree. Let $\sqcup_n S_n = \omega_1$ be a decomposition of ω_1 into stationary subsets. The CS-iteration of proper forcings $\langle P_i, \dot{Q}_j : i \leq \omega, j < \omega \rangle$ such that $\Vdash_{P_i} \dot{Q}_i = Q(T, S_i)$ satisfies the property that $\Vdash_{P_i} T$ is Baire for $i < \omega$ but $\Vdash_{P_\omega} T$ is special.

Semi-strongly proper forcings

Definition (Shelah)

A poset P is semi-strongly proper if for sufficiently large regular λ, for any $M \prec H(\lambda)$ containing P, for any countable sequence of dense subsets $\langle D_n : n \in \omega \rangle$ of $P \cap M$ and any $p \in P \cap M$, there exists $q \leq p$, such that for all $n \in \omega$, $q \Vdash D_n \cap \dot{G} \neq \emptyset$. We say such q is semi-strongly generic for M and $\langle D_n : n \in \omega \rangle$ (or just $\langle D_n : n \in \omega \rangle$ if M is clear from the context). Note that we don't require $D_n = D \cap M$ for some $D \in M$.

Lemma

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There are at least two proofs. Here is the “cheesy” one: for any Baire tree T and any semi-strongly proper P, $\Vdash_T P$ is semi-strongly proper, hence by the Baire preservation lemma, $\Vdash_P T$ is Baire.

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Theorem (Shelah)

CS-iteration of s.s.p forcings is s.s.p.

Hence we get $CON(RC^b + MA_{\omega_1}(s.s.p))$ for free.

Still not good enough

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Definition

A proper poset P is Baire indestructible if for any Baire tree T, $\Vdash_T \check{P}$ is proper. We call this class *Baire Indestructibly Proper (BIP)*.

Remark

It is possible to have an improper P and a Baire tree T such that $\Vdash_T P$ is proper. However, the latter implies that in V for sufficiently large regular λ,
$\{M \in [H(\lambda)]^\omega : P$ is proper with respect to $M\}$ is stationary.

Preservation theorem for BIP forcings

Lemma

Let T be a Baire tree and $\langle P_i, \dot{Q}_j : i \leq \alpha, j < \alpha \rangle$ be a countable support iteration of proper forcings such that for each $i < \alpha$, $\Vdash_{T \times P_i} \dot{Q}_i$ is proper. Then $\Vdash_T P_\alpha$ is proper.

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Corollary

CS iteration of BIP forcings is BIP. Thus CS iteration of BIP preserves Baire trees.

Preservation theorem for BIP forcings

Illustration of the main idea of the proof of the Key Lemma using two-step iteration (there is an easier argument for this case, but this idea also works in the limit case).

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Illustration of the main idea of the proof of the Key Lemma using two-step iteration (there is an easier argument for this case, but this idea also works in the limit case).
Fix $R = P * \dot{Q}$, $M \prec H(\lambda)$ containing R and a countable collection C of dense subsets of either $R \cap M$ or $P \cap M$.

Definition (Shelah)

We say C is *closed under operations* if for any $D \in C$ such that D is a dense subset of $R \cap M$ and any $(p, \dot{q}) \in M \cap R$,
$A_{D,(p,\dot{q})} = \{r \in P \cap M : r \perp p \vee \exists \dot{q}' \; r' =_{def} (r, \dot{q}') \in D, r' \leq (p, \dot{q})\}$
is also in the collection.

Preservation theorem for BIP forcings

Illustration of the main idea of the proof of the Key Lemma using two-step iteration (there is an easier argument for this case, but this idea also works in the limit case).
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Let $C_0 \subset C$ be the collection of dense subsets of $P \cap M$, $C_1 \subset C$ be the corresponding one for $R \cap M$. For any generic $G \subset P$ and any $D \in C_1$, let $(D)^G$ denote $\{(\dot{q})^G : \exists p \in G\ (p, \dot{q}) \in D\}$.

Preservation theorem for BIP forcings

Assume C is closed under operations.

Lemma (Shelah)

Fix some $q \in P$ that is semi-strongly generic for M and C_0, $q \Vdash_{P_\gamma} \dot{Q}$ is semi-strongly proper for $M[\dot{G}]$ and $(C_1)^{\dot{G}} =_{def} \{(D)^{\dot{G}} : D \in C_1\}$. Then there exists $\dot{r}$ such that $(q, \dot{r})$ is semi-strongly generic for M and C_1.

Key lemma in the simplified scenario

Sketch of the Key Lemma:
Let $H \subset T$ be generic over V. Let λ be a sufficiently large regular cardinal containing $R = P * \dot{Q}$ and other relevant objects such that $M' = M \cap H(\lambda)^V \prec H(\lambda)^V$.

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Let $H \subset T$ be generic over V. Let λ be a sufficiently large regular cardinal containing $R = P * \dot{Q}$ and other relevant objects such that $M' = M \cap H(\lambda)^V \prec H(\lambda)^V$. Let C_0 be the collection of $D \cap M = D \cap M'$ where $D \in M$ is a dense subset of P, and C_1 be the collection of $D \cap M = D \cap M'$ where $D \in M$ is a dense subset of R. **Notice** $C_0, C_1 \in V$ and $C_0 \cup C_1$ is closed under operations.

Claim

In V, P is semi-strongly generic with respect to M' and C_0.

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Sketch.

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Claim

In V, for any $q \in P$ that is semi-strongly generic for M' and C_0, $q \Vdash_P \dot{Q}$ is semi-strongly proper for $M'[\dot{G}]$ and $(C_1)^{\dot{G}}$.

Sketch.

Use the fact that in $V[H]$, $\Vdash_P \dot{Q}$ is proper for $M[\dot{G}]$. □

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In V, P is semi-strongly generic with respect to M' and C_0.

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Finally, we use Shelah's lemma in V to see that $R = P * \dot{Q}$ is semi-strongly proper for M' and C_1. This implies that in $V[H]$, R is proper for M.

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