Big Ramsey degrees and forbidden cycles

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Definition

Let **A** be a countable relational structure. We say that **A** has *finite big Ramsey degrees* if for every $n \in \omega$ there is $D(n) \in \omega$ such that for every finite coloring of $[\mathbf{A}]^n$ there is a copy **B** of **A** (inside of **A**) such that $[\mathbf{B}]^n$ has at most $D(n)$ colors.

Example

- $(\omega,$ no structure$)$ (Ramsey)
- $(\mathbb{Q}, <)$ (Galvin, Laver, Devlin)
- Random (Rado) graph (Todorčević, Sauer)
- Triangle free Henson graph $\mathbb{H}_3$ (Dobrinen, Hubička)

Definition

A structure **A** is universal for a class of structures $\mathcal{C}$ if **A** contains a copy of every $\mathbf{B} \in \mathcal{C}$.

Proposition

If $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ are both universal for $\mathcal{C}$ and **A** has finite big Ramsey degrees, then **B** also has finite big Ramsey degrees.

Words and parametric spaces

Let Σ be a finite alphabet A word U of length n in Σ is a sequence of letters $U\colon n \to \Sigma$.
The set of all finite words in Σ is denoted Σ^*.

A parameter word over empty alphabet with parameters $\langle \lambda_i, i \in \omega \rangle$ is a sequence $x\colon \omega \to \{ \lambda_i, i \in \omega \}$ such that each λ_i is in the range of x and for $i < j$ the first occurrence of λ_i is before the first occurrence of λ_j. We write $x \in [\emptyset]\binom{\omega}{\omega}$

Let $x \in [\emptyset]\binom{\omega}{\omega}$. The substitution function $S_x\colon \Sigma^* \to \Sigma^*$ is defined as follows. Suppose $U \in \Sigma^*$ of length n. For $i < n$ replace each occurrence of λ_i in x by $U(i)$, and truncate at the first occurrence of λ_n. The result is the word $S_x(U)$.

Big Ramsey degrees using parameter spaces

Jan Hubička, *Big Ramsey degrees using parameter spaces*, https://arxiv.org/abs/2009.00967

Let $\mathcal{C}$ be a class of countable relational structures. To prove that universal structures in $\mathcal{C}$ have finite big Ramsey degrees it is sufficient to do the following:

1. Choose a suitable finite alphabet Σ.
2. Find $\mathbf{A} \in \mathcal{C}$, $\mathbf{A} = (\Sigma^*, R)$ such that $\mathbf{A}$ is universal.
3. Check that for every $x \in [\emptyset]\binom{\omega}{\omega}$ the substitution map $S_x : \Sigma^* \to \Sigma^*$ is a structural embedding of $\mathbf{A}$ into itself.

Then the Carlson–Simpson theorem implies the desired conclusion.

Example – graphs

$\Sigma = \{\mathrm{e}, \mathrm{n}\}$

For $U, V \in \Sigma^*$, $|U| < |V|$ declare that $(U, V) \in E$ iff $V(|U|) = \mathrm{e}$.
This is called the *passing number representation*.

U	e	e	n	e	n	e			
V	n	n	n	e	e	n	e	e	
W	n	e	e	n	e	n	n	n	e

Here $(U, V) \in E$, $(V, W) \in E$, $(U, W) \notin E$.

Lemma

(Σ^*, E) is a universal countable graph.

Lemma

For every $x \in [\emptyset]\binom{\omega}{\omega}$ the substitution map $S_x \colon \Sigma^* \to \Sigma^*$ is a graph embedding.

Main Theorem

Let L be a finite language consisting of unary and binary symbols, and let $\mathbf{K}$ be a countably-infinite irreducible structure. Assume that every countable structure $\mathbf{A}$ has a completion to $\mathbf{K}$ provided that every induced cycle in $\mathbf{A}$ (seen as a substructure) has a completion to $\mathbf{K}$ and every irreducible substructure of $\mathbf{A}$ of size at most 2 embeds into $\mathbf{K}$. Then $\mathbf{K}$ has finite big Ramsey degrees.

D-metric space

$D = \{ 1, 2, 3, \ldots, \mathfrak{d} \}$ set of distances.
We write $\Delta(i, j, k)$ if $i, j, k \in D$ fulfill the triangle inequality (i.e. $i + j \geq k$, etc.).

A D-metric space is a pair (A, d) where $d\colon [A]^2 \to D$ such that $\Delta(d(a, b), d(b, c), d(a, c))$ for each $a, b, c \in A$.

A partial D-metric space is a pair (B, d) where $d; [A]^2 \to D$ is a partial function and there exists $\bar{d} \supseteq d$ such that $(B, \bar{d})$ is a D-metric space. We call the space $(B, \bar{d})$ a completion of the partial space (B, d).

Proposition

Let $d\colon [A]^2 \to D$ be a function. Then (A, d) is a partial D-metric space iff every induced cycle in (A, d) is a partial D-metric space.

Theorem

Universal countable D-metric spaces have finite big Ramsey degrees.

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Universal countable D-metric spaces have finite big Ramsey degrees.

Let $\Sigma = D$.

Define (Σ^*, d) as follows. Suppose $U, V \in \Sigma^*$.

If $|U| < |V|$ and $\Delta(U(i), V(i), V(|U|))$ for each $i < |U|$ then let $d(U, V) = V(|U|)$.

Otherwise leave $d(U, V)$ undefined.

U	2	2	4	…	4	1	3	5	2	4		
V	3	4	3	…	2	1	4	3	5	2	4	…

Lemma

(Σ^*, d) *is a universal structure for D-metric spaces.*

Lemma

(Σ^*, d) *is a partial D-metric space.*

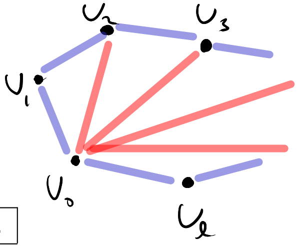
Lemma

(Σ^*, d) *is a partial D-metric space.*

Proof.

Suppose $C = \langle U_0, U_1, \ldots, U_\ell \rangle$ is an induced cycle in (Σ^*, d). Suppose U_0 is the shortest word in the cycle.

U_0	2	3	1	...	4					
U_1	5	6	3	...	2	4	3	5		
U_2	2	5	3	...	4	3				
U_3	3	4	3	...	4	3	5	2	4	
...										
U_ℓ	3	4	3	...	4	3	5	2	4	2



For $k \in \{2, 3, \ldots, \ell - 1\}$ define $e(U_0, U_k) = U_k(|U_0|)$.
Notice that all triangles is $(C, d \cup e)$ are metric, and it a partial D-metric space.