Some applications of covering matrices

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This talk includes joint work with Šárka Stejskalová and with Assaf Rinot.

I : Basic definitions

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Definition (Viale)

Suppose that $\mathcal{D}$ is a covering matrix for λ^+. We say that $\mathrm{CP}(\mathcal{D})$ holds if there is an unbounded $A \subseteq \lambda^+$ such that, for all $x \in [A]^\theta$, there are $i < \theta$ and $\beta < \lambda^+$ such that $x \subseteq D(i, \beta)$.

II : Cardinal arithmetic

Meeting numbers

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For cardinals $\theta < \lambda$, the meeting number $m(\theta, \lambda)$ is the minimal size of a family $\mathcal{X} \subseteq [\lambda]^\theta$ such that, for all $y \in [\lambda]^\theta$, there is $x \in \mathcal{X}$ such that $|x \cap y| = \theta$.

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III: Tightness of G_δ-modifications

G_δ-modifications

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4. The *G_δ-modification* of X, denoted by X_δ, is the space with the same underlying set as X and with a base consisting of the G_δ sets of X.

Background results

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Theorem (Usuba)

There is a normal countably tight space X such that $t(X_\delta) > 2^\omega$.

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Suppose that κ is a regular uncountable cardinal and $\square(\kappa)$ holds. Then there is a Fréchet, α_1-space X such that $t(X_\delta) = \kappa$.

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3. *if SCH fails at λ, i.e., if λ is strong limit and $\lambda^\omega > \lambda^+$, then X is Fréchet.*

Corollary

If SCH *fails at* λ*, then there is a Fréchet,* α_1*-space* X *such that* $t(X_\delta) = \lambda^+$*.*

IV: Coloring numbers

Coloring numbers of graphs

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Can we have G as in the corollary with $\mathrm{Col}(G) = \theta^{++}$? The simplest arrangement in which this could conceivably happen is $\kappa = \aleph_{\omega+1}$ and $\theta = \aleph_0$.

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Conjecture

For all infinite cardinals $\theta < \kappa$*, if* $G = (\kappa, E)$ *is a graph such that* $\mathrm{Col}(G_\alpha) \leq \theta$ *for all* $\alpha < \kappa$*, then* $\mathrm{Col}(G) \leq \theta^+$*.*

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Lemma (Todorcevic)

If λ is singular and $\square_\lambda$ holds, then there is a nice covering matrix $\mathcal{D}$ for λ^+ such that, for all $\alpha < \beta < \lambda^+$ and all $j < \mathrm{cf}(\lambda)$, there is $i < \mathrm{cf}(\lambda)$ such that $D(j, \beta) \cap \alpha \subseteq D(i, \alpha)$.

Sketch of proof.

Suppose for a contradiction that $\mathrm{cf}(\lambda) = \theta < \lambda$, $\Box_\lambda$ holds, and $G = (\lambda^+, E)$ is a graph such that $\mathrm{Col}(G) = \theta^{++}$ but $\mathrm{Col}(G \restriction \alpha) \leq \theta$ for all $\alpha < \lambda^+$.

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But $|D(i, \eta)| < \lambda$, and there are unboundedly many $\beta < \lambda^+$, each of which is connected to at least θ-many elements of $D(i, \eta)$. This immediately yields an initial segment of G with coloring number greater than θ. $\square$

Thank you for your attention!