Baumgartner's Isomorphism Theorem for a Kurepa Line

Hossein Lamei Ramandi

University of Münster

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Introduction

Recall the following theorem from the 70's.

Theorem (Baumgartner)

It is consistent with ZFC that every two $\aleph_1$-dense subsets of the reals are order isomorphic.

• A linear order L without endpoints is said to be κ-dense if for all $x < y$ in L there are exactly κ many elements in between x, y.

• Baumgartner viewed his theorem, which is a consequence of PFA, as an extension of Cantor's famous theorem on the rationals: every two $\aleph_0$-dense linear orders are isomorphic.

• Assume L is a linear order. Density of L is $\min\{|D| : D \subset L \text{ is dense }\}$.

• The analogue of Baumgartner's theorem for homogeneous linear orders of density $\aleph_1$ is closely related to Kurepa lines.

• A linear order is said to be homogeneous if every two non-empty open intervals are isomorphic.

Kurepa lines

Definition

A linear order is said to be Kurepa if

- $|L| \geq \aleph_2$,
- the closure of any countable subset of L is countable, and
- density of L is $\aleph_1$

• There is a Kurepa line if and only if there is a Kurepa tree.
• If there is an inaccessible cardinal then it is consistent that there are no Kurepa trees (lines).

An isomorphism theorem for Kurepa lines

Theorem

It is consistent with CH there is a homogeneous Kurepa line K of size $\aleph_2$ such that whenever $L \subset K$ is $\aleph_2$-dense and all $x \in L$ have uncountable cofinality and coinitiaity, then $L \simeq K$.

Some technical terminology

Definition

Assume X is uncountable, $S \subset [X]^\omega$ and P is a forcing. We say that P is S-complete if for all suitable models M with $M \cap X \in S$ all decreasing (M, P)-generic ω-sequences of elements of P have a lower bound in P.

• This criteria is used in order to show certain sets remain stationary and ω_1 is preserved.

• In order to make sure $\aleph_2$ is preserved by our forcings, we use ideas behind Shelah's proper isomorphism condition machinery.

Generating a candidate for K

Definition

Q is the poset consisting of all conditions $q = (T_q, b_q)$ such that

- T_q is a countable lexicographically ordered tree of height $\alpha_q + 1$ such that every element of it has an extension in the top level, and
- b_q is a countable partial function from ω_2 onto the top level of T_q.

We let $q \leq p$ if:

- T_q is an end-extension of T_p,
- $dom(b_q) \supset dom(b_p)$,
- for all $\xi \in dom(b_p)$, $b_p(\xi) \leq_{T_q} b_q(\xi)$

Regarding Q...

• It is well known that Q is countably closed and under CH, has the $\aleph_2$-chain condition.
• Assume $G \subset Q$ is generic.
• From now on $T = \bigcup\{T_q : q \in G\}$.
• $b_\xi = \{b_q(\xi) : q \in G\}$ is a cofinal branch of T.
•$B(T) = \{b_\xi : \xi \in \omega_2\}$ is the set of all branches of T.
•$\Omega(T)$ is stationary in $[B(T)]^\omega$.

Adding tree embeddings to T

Definition

Suppose T is as above U a downward closed everywhere Kurepa subtree of T, and $C_U \subset \omega_2$ a club that is fast enough. The poset $Q_{T,U}$ is the set of all conditions $p = (f_p, \phi_p)$ such that

1. $f_p : T \restriction A_p \longrightarrow U \restriction Ap$ is a level preserving tree isomorphism, where $A_P \subset \omega_1$ is countable and closed with max$A_p = \alpha_p$,
2. ϕ_p is a countable partial injection on ω_2 such that
    1. for all $\xi \in dom(\phi_p), b_{\phi_p}(\xi) \in B(U)$,
    2. ϕ_p respects C_U
3. if $dom(\phi_p)$ then $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)(\alpha_p)}$.

We let $p \leq q$ if $f_q \subset f_p$, $f_p \restriction T_q = f_q$ and $\phi_p \subset \phi_q$.

This forcing is $\Omega(T)$-complete and satisfies the $\Omega(T)$-completeness isomorphism condition.

Fast club definition

Definition

A club $C \subset \omega_2$ is fast enough for U and T if it is the set of all $\sup(M_\xi \cap \omega_2)$ where $\langle M_\xi : \xi \in \omega_2 \rangle$ is a continuous $\in$-chain of $\aleph_1$-sized elementary submodels of H_θ such that

- U, T are in M_0
- for all $\xi \in \omega_1$, $\xi \cup \omega_1 \subset M_\xi$
- for all $\xi \in \omega_1$, the sequence up to ξ is in $M_{\xi+1}$

Let ϕ be a partial function on ω_2 and $C \subset \omega_2$. We say ϕ respects C whenever $\xi < \alpha$ is equivalent to $\phi(\xi) < \alpha$ for all $\xi \in dom(\phi)$ and $\alpha \in C$.

Making small dense sets biembeddable

Definition

Assume T is as above and X, Y are two subsets of ω_2 such that $|X| = |Y| = \aleph_1$ and the closure of both $\{b_\xi : \xi \in X\}$, $\{b_\xi : \xi \in Y\}$ have cardinality $\aleph_2$. Let $U = \bigcup\{b_\xi : \xi \in X\}$ and $V = \bigcup\{b_\xi : \xi \in Y\}$. $\mathcal{F}_{XY}(= \mathcal{F})$ is the poset consisting of all conditions $p = (f_p, \phi_p)$ for which the following hold:

1. $f_p : U \restriction A_p \longrightarrow V \restriction A_p$ is a lex order and level preserving tree isomorphism where $A_p \subset \omega_1$ is countable and closed with $\max A_p = \alpha_p$.
2. ϕ_p is a countable partial injection from ω_2 to ω_2 such that:
    1. ϕ_p respects a fast enough club,
    2. for all $\xi \in dom(\phi_p)$, if $\xi \in X$ then $\phi_p(\xi) \in Y$, and
    3. the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lexicographic order preserving.
3. For all $t \in T_{\alpha_p}$ there are at most finitely many $\xi \in (\phi_p)$ with $t \in b_\xi$.
4. For all $\xi \in (\phi_p)$, $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p)$.

We let $q \leq p$ if $f_p \subset f_q$, $A_q \cap \alpha_p = A_p$ and $\phi_p \subset \phi_q$.

This forcing is $\Omega(T)$-complete but it does not satisfy $\Omega(T)$-cic.

By iterating these posets over a model of GCH we can get reach a model in which the following hold.

1. T is Kurepa.
2. T is club isomorphic to all of its everywhere Kurepa subtrees and has no Aronszajn subtree.
3. $\Omega(T)$ is stationary.
4. If X, Y are two suborders of $K = B(T)$ and $|X| = |Y| = \aleph_1$ and their closure has cardinality $\aleph_2$ then X embeds into Y as a linear order.

K can be a minimal Kurepa line

For all $\aleph_2$-dense $L \subset K$ we add an $\aleph_2$-dense closed $Z \subset L$.

Definition

Assume L is as above. P_L is the set of all conditions $p = (Z_p, A_p)$ such that:

- $Z_p \subset L$ is countable and non-empty,
- A_p is a countable antichain of T such that $Z_p \cap A_p = \emptyset$.

- P_L is $\Omega(T)$-complete and has the $\Omega(T)$-cic. In particular preserves $\aleph_1, \aleph_2$.
- Assume $G \subset P_L$ is generic. Then $\bigcup_{p \in G} Z_p \cup \{b \in K : \exists p \in G\ b \cap A_p \neq \emptyset\} = K$.
- After forcing with P_L, $2^{\omega_1} = 2^{\omega_2}$.

By iterating these posets over a model of GCH we can get reach a model in which the following hold.

1. T is Kurepa.
2. T is club isomorphic to all of its everywhere Kurepa subtrees and has no Aronszajn subtree.
3. $\Omega(T)$ is stationary.
4. If X, Y are two suborders of $K = B(T)$ and $|X| = |Y| = \aleph_1$ and their closure has cardinality $\aleph_2$ then X embeds into Y as a linear order.
5. K embeds in all of its $\aleph_2$-dense suborders.

Isomorphism Forcings

- $L \subset K$ is $\aleph_2$-dense and there is no $x \in L$ with countable cofinality or coinitiality.
- If $t \in T$, L_t refers to the set of all $b \in K$ such that $t \in b$.

Definition

Assume L is as above and $U = \bigcup L$. For each $(t, s) \in T \otimes U$ fix embeddings $i_{ts} : K_t \longrightarrow L_s$ and $j_{st} : L_s \longrightarrow K_t$. The forcing I consists ofconditions $p = (f_p, \phi_p)$ such that:

- $f_p : T \restriction A_p \longrightarrow U \restriction A_p$ is a lex order and level preserving tree isomorphism where $A_p \subset \omega_1$ is countable and closed with $\max A_p = \alpha_p$.
- ϕ_p is a countable partial injection from K to L such that for all $b \in dom(\phi_p)$ there $(t, s) \in T \otimes U$ such that $\phi_p(b) = i_{ts}(b)$ or $\phi_p(b) = j_{st}(b)$.
- For each $t \in T_{\alpha_p}$ there are at most finitely many $b \in K$ with $t \in b \in dom(\phi_p)$.
- for all $b \in dom(\phi_p)$, $f_p(b(\alpha_p)) = [\phi_p(b)](\alpha_p)$.

This poset is $\Omega(T)$-complete and has the $\Omega(T)$-cic.