On the resolvability of product spaces

István Juhász

Alfréd Rényi Institute of Mathematics

SETTOP 2022, Novi Sad, August 22 - 25, 2022

Introduction

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \;\Leftrightarrow\; A \in \mathcal{N}(X)$ for any $A \subset X$.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $Int\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

X is maximally resolvable if it is $\Delta(X)$-resolvable, where
$\Delta(X) = \min\{|U| : U \in \tau^+(X)\}$.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $\mathit{Int}\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

X is maximally resolvable if it is $\Delta(X)$-resolvable, where
$\Delta(X) = \min\{|U| : U \in \tau^+(X)\}$.
FACT. Finite spaces are maximally resolvable.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $Int\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

X is maximally resolvable if it is $\Delta(X)$-resolvable, where
$\Delta(X) = \min\{|U| : U \in \tau^+(X)\}$.
FACT. Finite spaces are maximally resolvable.
So, in what follows, for any X we assume $|U| \geq \omega$ for all $U \in \tau^+(X)$.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $Int\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \geq \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

X is maximally resolvable if it is $\Delta(X)$-resolvable, where
$\Delta(X) = \min\{|U| : U \in \tau^+(X)\}$.
FACT. Finite spaces are maximally resolvable.
So, in what follows, for any X we assume $|U| \geq \omega$ for all $U \in \tau^+(X)$.
All crowded T_0-spaces are such.

Introduction

A space X is κ-resolvable if it has κ pairwise disjoint dense sets.
κ-irresolvable otherwise.
(ir)resolvable = 2-(ir)resolvable)

X is OHI if every $U \in \tau^+(X)$ is irresolvable.
FACT. If X is irresolvable then there is an OHI $U \in \tau^+(X)$.
If X is OHI then $Int\, A = \emptyset \Leftrightarrow A \in \mathcal{N}(X)$ for any $A \subset X$.

X is neat if $|U| = |X|$ for every $U \in \tau^+(X)$.
Clearly, $\{U \in \tau^+(X) : U \text{ is neat}\}$ is a π-base of X.
FACT. For every $\kappa \ge \omega$ there is a 0-dimensional T_2-space X of size κ that is both OHI and neat. (Hewitt, 1943 for $\kappa = \omega$.)

X is maximally resolvable if it is $\Delta(X)$-resolvable, where
$\Delta(X) = \min\{|U| : U \in \tau^+(X)\}$.
FACT. Finite spaces are maximally resolvable.
So, in what follows, for any X we assume $|U| \ge \omega$ for all $U \in \tau^+(X)$.
All crowded T_0-spaces are such.

Part I: irresolvable products

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973:

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\}\rangle$ with any countable irresolvable space is irresolvable.

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

M1 Can T_1 be improved to T_2 or T_3?

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?
(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?
- M3 Can both factors be "small"?

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?
- M3 Can both factors be "small"?
- M4 Can we find three spaces whose product is irresolvable?

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?
- M3 Can both factors be "small"?
- M4 Can we find three spaces whose product is irresolvable?

FACT. Any product of infinitely many non-singleton spaces is $\mathfrak{c}$-resolvable.

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?
- M3 Can both factors be "small"?
- M4 Can we find three spaces whose product is irresolvable?

FACT. Any product of infinitely many non-singleton spaces is $\mathfrak{c}$-resolvable.

Proof: For $x, y \in \prod\{X_n : n < \omega\}$ let $x \sim y$ iff $|\{n : x(n) \neq y(n)\}| < \omega$.

Part I: irresolvable products

El'kin: Are there two spaces whose product is irresolvable?

(He proved: X is irresolvable iff $\tau^+(X)$ contains an ultrafilter base.)

Malychin, 1973: If $\mathcal{U}$ is a σ-complete free ultrafilter on κ then the product of the T_1-space $\langle \kappa, \mathcal{U} \cup \{\emptyset\} \rangle$ with any countable irresolvable space is irresolvable.

- M1 Can T_1 be improved to T_2 or T_3?
- M2 Is the measurable cardinal necessary?
- M3 Can both factors be "small"?
- M4 Can we find three spaces whose product is irresolvable?

FACT. Any product of infinitely many non-singleton spaces is $\mathfrak{c}$-resolvable.

Proof: For $x, y \in \prod\{X_n : n < \omega\}$ let $x \sim y$ iff $|\{n : x(n) \neq y(n)\}| < \omega$.
Every $\sim$-equivalence class is dense in the product.

Part I: irresolvable products 2.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*,

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.

$MR(X) = \{\kappa : X$ is monotonically κ-resolvable$\}$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.

$MR(X) = \{\kappa : X \text{ is monotonically } \kappa\text{-resolvable}\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X$ is monotonically κ-resolvable$\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X \text{ is monotonically } \kappa\text{-resolvable}\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X \text{ is monotonically } \kappa\text{-resolvable}\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

THEOREM

If $X \times Y$ is irresolvable then $MR(X) \cap MR(Y) = \emptyset$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X$ is monotonically κ-resolvable$\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

THEOREM

If $X \times Y$ is irresolvable then $MR(X) \cap MR(Y) = \emptyset$.

PROOF. If $\kappa \in MR(X) \cap MR(Y)$ then there are monotone κ-resolutions $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ and $Y = \bigcup\{B_\alpha : \alpha < \kappa\}$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X \text{ is monotonically } \kappa\text{-resolvable}\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

THEOREM

If $X \times Y$ is irresolvable then $MR(X) \cap MR(Y) = \emptyset$.

PROOF. If $\kappa \in MR(X) \cap MR(Y)$ then there are monotone κ-resolutions $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ and $Y = \bigcup\{B_\alpha : \alpha < \kappa\}$.
Let $rk_X(x) = \min\{\alpha : x \in A_\alpha\}$ and $rk_Y(y) = \min\{\alpha : y \in B_\alpha\}$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X$ is monotonically κ-resolvable$\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

THEOREM

If $X \times Y$ is irresolvable then $MR(X) \cap MR(Y) = \emptyset$.

PROOF. If $\kappa \in MR(X) \cap MR(Y)$ then there are monotone κ-resolutions $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ and $Y = \bigcup\{B_\alpha : \alpha < \kappa\}$.
Let $rk_X(x) = \min\{\alpha : x \in A_\alpha\}$ and $rk_Y(y) = \min\{\alpha : y \in B_\alpha\}$.
Then $\{\langle x, y\rangle : rk_X(x) \leq rk_Y(y)\}$ and $\{\langle x, y\rangle : rk_X(x) > rk_Y(y)\}$ are disjoint dense sets in $X \times Y$.

Part I: irresolvable products 2.

Unlabeled items are from: J-Soukup-Szentmiklóssy, *On the resolvability of products*, https://arxiv.org/pdf/2205.14896.pdf

DEFINITION. X is *monotonically κ-resolvable* if $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ for $\{A_\alpha : \alpha < \kappa\}$ increasing s.t. $Int(A_\alpha) = \emptyset$ for all $\alpha < \kappa$.
$MR(X) = \{\kappa : X$ is monotonically κ-resolvable$\}$.

EXAMPLES. (i) If $\kappa \geq \omega$ and X is κ-resolvable then $\kappa \in MR(X)$.
(ii) If X is neat then $|X| \in MR(X)$.
(iii) If X is meager then $\omega \in MR(X)$.

THEOREM

If $X \times Y$ is irresolvable then $MR(X) \cap MR(Y) = \emptyset$.

PROOF. If $\kappa \in MR(X) \cap MR(Y)$ then there are monotone κ-resolutions $X = \bigcup\{A_\alpha : \alpha < \kappa\}$ and $Y = \bigcup\{B_\alpha : \alpha < \kappa\}$.
Let $rk_X(x) = \min\{\alpha : x \in A_\alpha\}$ and $rk_Y(y) = \min\{\alpha : y \in B_\alpha\}$.
Then $\{\langle x, y\rangle : rk_X(x) \leq rk_Y(y)\}$ and $\{\langle x, y\rangle : rk_X(x) > rk_Y(y)\}$ are disjoint dense sets in $X \times Y$.

Part I: irresolvable products 3.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space,

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^{+}(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

COROLLARY

If X is neat then X^2 is resolvable.

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

COROLLARY

If X is neat then X^2 is resolvable. (Equivalently: the product of two neat spaces of the same size is resolvable.)

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

COROLLARY

If X is neat then X^2 is resolvable. (Equivalently: the product of two neat spaces of the same size is resolvable.)

QUESTION. Can we do better? E.g., is X^2 always 3-resolvable?

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

COROLLARY

If X is neat then X^2 is resolvable. (Equivalently: the product of two neat spaces of the same size is resolvable.)

QUESTION. Can we do better? E.g., is X^2 always 3-resolvable?

Part I: irresolvable products 3.

COROLLARY (Bešlagić and Levy, 1996)

If $X \times Y$ is irresolvable then there is an irresolvable Bare-space, hence there is an inner model with a measurable cardinal.

PROOF. We may assume that $X \times Y$ is OHI, hence so are both X and Y. Also, $\omega \notin MR(X) \cap MR(Y)$, so X or Y is non-meager, say X is. It's known that then there is $U \in \tau^+(X)$ that is Baire. The rest follows from Kunen-Szymanski-Tall.

This answers question (M2).

COROLLARY

If X is neat then X^2 is resolvable. (Equivalently: the product of two neat spaces of the same size is resolvable.)

QUESTION. Can we do better? E.g., is X^2 always 3-resolvable?

Part I: irresolvable products 4.

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if $X,\ Y$ are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable.

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if X, Y are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if $X,\ Y$ are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if X, Y are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if $X,\ Y$ are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if X, Y are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

THEOREM (JSSz)

X^2 is ω-resolvable for any countable (crowded) T_3-space X.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if $X,\ Y$ are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

THEOREM (JSSz)

X^2 is ω-resolvable for any countable (crowded) T_3-space X.

This does use topology!

Part I: irresolvable products 4.

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if X, Y are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

THEOREM (JSSz)

X^2 is ω-resolvable for any countable (crowded) T_3-space X.

This does use topology!

PROBLEM. How about T_2-spaces?

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if $X,\ Y$ are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

THEOREM (JSSz)

X^2 is ω-resolvable for any countable (crowded) T_3-space X.

This does use topology!

PROBLEM. How about T_2-spaces?

We dont even know if X^2 is 3-resolvable!

DEFINITION. $S(\kappa, \lambda, \mu)$ says that there is a coloring $c : \kappa \times \lambda \to \mu$ s.t. $c[A \times B] = \mu$ for any $A \in [\kappa]^\kappa$ and $B \in [\lambda]^\lambda$.

$S(\kappa, \lambda, \mu)$ implies that if X, Y are neat with $|X| = \kappa$ and $|Y| = \lambda$ then $X \times Y$ is μ-resolvable. We have $S(\kappa, \kappa, 2)$ for any $\kappa \geq \omega$!

FACT. $S(\lambda^+, \lambda^+, \lambda^+)$ holds if λ is regular.
(This doesn't work for ω or weakly compacts.)
Also, no topology is involved!

THEOREM (JSSz)

X^2 is ω-resolvable for any countable (crowded) T_3-space X.

This does use topology!

PROBLEM. How about T_2-spaces?

We dont even know if X^2 is 3-resolvable!

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$. By transfinite recursion define $B_\alpha \in \mathcal{B}$, $i_\alpha < 2$ for $\alpha < \mu$ s.t. $\{B_\alpha : \alpha < \mu\} \subset \mathcal{B}$ is decreasing and $B_\alpha \subset Z_{y_\alpha, i_\alpha}$, i.e. $B_\alpha \times \{y_\alpha\} \subset Z_{i_\alpha}$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$. By transfinite recursion define $B_\alpha \in \mathcal{B}$, $i_\alpha < 2$ for $\alpha < \mu$ s.t. $\{B_\alpha : \alpha < \mu\} \subset \mathcal{B}$ is decreasing and $B_\alpha \subset Z_{y_\alpha, i_\alpha}$, i.e. $B_\alpha \times \{y_\alpha\} \subset Z_{i_\alpha}$. By $\mu < \lambda$ there is $B \in \mathcal{B}$ s.t. $B \subset \bigcap\{B_\alpha : \alpha < \mu\}$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$. By transfinite recursion define $B_\alpha \in \mathcal{B}$, $i_\alpha < 2$ for $\alpha < \mu$ s.t. $\{B_\alpha : \alpha < \mu\} \subset \mathcal{B}$ is decreasing and $B_\alpha \subset Z_{y_\alpha, i_\alpha}$, i.e. $B_\alpha \times \{y_\alpha\} \subset Z_{i_\alpha}$. By $\mu < \lambda$ there is $B \in \mathcal{B}$ s.t. $B \subset \bigcap\{B_\alpha : \alpha < \mu\}$.
Let $A_i = \{y_\alpha : i_\alpha = i\}$ for $i < 2$, then $Y = A_0 \cup A_1$, so $V = Int(A_i) \neq \emptyset$ for some $i < 2$.

Part I: irresolvable products 5.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$. By transfinite recursion define $B_\alpha \in \mathcal{B}$, $i_\alpha < 2$ for $\alpha < \mu$ s.t. $\{B_\alpha : \alpha < \mu\} \subset \mathcal{B}$ is decreasing and $B_\alpha \subset Z_{y_\alpha, i_\alpha}$, i.e. $B_\alpha \times \{y_\alpha\} \subset Z_{i_\alpha}$. By $\mu < \lambda$ there is $B \in \mathcal{B}$ s.t. $B \subset \bigcap\{B_\alpha : \alpha < \mu\}$.
Let $A_i = \{y_\alpha : i_\alpha = i\}$ for $i < 2$, then $Y = A_0 \cup A_1$, so $V = Int(A_i) \neq \emptyset$ for some $i < 2$. For any $y_\alpha \in V$ we have $B \times \{y_\alpha\} \subset B_\alpha \times \{y_\alpha\} \subset Z_i$, hence $B \times V \subset Z_i$.

How to get irresolvable products with "nice and small" factors?

DEFINITION. $\mathcal{S}$ is *λ-closed* if for any $\alpha < \lambda$ and *monotone decreasing* $\{S_\beta : \beta < \alpha\} \subset \mathcal{S}$ there is $S \in \mathcal{S}$ s.t. $S \subset \bigcap\{S_\beta : \beta < \alpha\}$.

THEOREM

If the OHI space X has a λ-closed π-base $\mathcal{B}$ then $X \times Y$ is irresolvable whenever Y is irresolvable and $|Y| < \lambda$.

PROOF. Let $X \times Y = Z_0 \cup Z_1$. For $y \in Y$ and $i < 2$ let $Z_{y,i} = \{x \in X : \langle x, y\rangle \in Z_i\}$. Then $X = Z_{y,0} \cup Z_{y,1}$ and so $Int(Z_{y,0}) \cup Int(Z_{y,1})$ is dense open in X by OHI.
Fix $Y = \{y_\alpha : \alpha < \mu\}$. By transfinite recursion define $B_\alpha \in \mathcal{B}$, $i_\alpha < 2$ for $\alpha < \mu$ s.t. $\{B_\alpha : \alpha < \mu\} \subset \mathcal{B}$ is decreasing and $B_\alpha \subset Z_{y_\alpha, i_\alpha}$, i.e. $B_\alpha \times \{y_\alpha\} \subset Z_{i_\alpha}$. By $\mu < \lambda$ there is $B \in \mathcal{B}$ s.t. $B \subset \bigcap\{B_\alpha : \alpha < \mu\}$.
Let $A_i = \{y_\alpha : i_\alpha = i\}$ for $i < 2$, then $Y = A_0 \cup A_1$, so $V = Int(A_i) \neq \emptyset$ for some $i < 2$. For any $y_\alpha \in V$ we have $B \times \{y_\alpha\} \subset B_\alpha \times \{y_\alpha\} \subset Z_i$, hence $B \times V \subset Z_i$.

Part I: irresolvable products 6.

Part I: irresolvable products 6.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$ $B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.

Part I: irresolvable products 6.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$
$B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$
$B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.
(iii) The λ-closed $\mathcal{B}(\mathcal{S}) = \{B_p : p \in Fn(\mathcal{S}, 2; \lambda)\}$ is the base of a 0-dimensional T_2 topology $\tau(\mathcal{S})$ on κ that is irresolvable.

Part I: irresolvable products 6.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$
$B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.
(iii) The λ-closed $\mathcal{B}(\mathcal{S}) = \{B_p : p \in Fn(\mathcal{S}, 2; \lambda)\}$ is the base of a 0-dimensional T_2 topology $\tau(\mathcal{S})$ on κ that is irresolvable. So, there is $X(\mathcal{S}) \in \tau(\mathcal{S})^+$ that is OHI.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$
$B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.
(iii) The λ-closed $\mathcal{B}(\mathcal{S}) = \{B_p : p \in Fn(\mathcal{S}, 2; \lambda)\}$ is the base of a 0-dimensional T_2 topology $\tau(\mathcal{S})$ on κ that is irresolvable. So, there is $X(\mathcal{S}) \in \tau(\mathcal{S})^+$ that is OHI.

THEOREM (Kunen)

The existence of a measurable cardinal implies the consistency of $MI(\kappa, \omega_1)$ with $\kappa \leq 2^{\omega_1}$, moreover $MI(\kappa, \omega_1)$ implies CH.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is λ*-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$ $B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.
(iii) The λ-closed $\mathcal{B}(\mathcal{S}) = \{B_p : p \in Fn(\mathcal{S}, 2; \lambda)\}$ is the base of a 0-dimensional T_2 topology $\tau(\mathcal{S})$ on κ that is irresolvable. So, there is $X(\mathcal{S}) \in \tau(\mathcal{S})^+$ that is OHI.

THEOREM (Kunen)

The existence of a measurable cardinal implies the consistency of $MI(\kappa, \omega_1)$ with $\kappa \leq 2^{\omega_1}$, moreover $MI(\kappa, \omega_1)$ implies CH.

COROLLARY

The existence of a measurable cardinal implies the consistency of having a 0-dimensional T_2-space of size $\leq 2^{\omega_1}$ whose product with any countable irresolvable space is irresolvable.

Part I: irresolvable products 6.

DEFINITION. (i) $\mathcal{S} \subset \mathcal{P}(\kappa)$ is *λ-independent* if for any $p \in Fn(\mathcal{S}, 2; \lambda)$ $B_p = \bigcap\{S : p(S) = 1\} \cap \bigcap\{\kappa \setminus S : p(S) = 0\} \neq \emptyset$.
(ii) $MI(\kappa, \lambda)$ says that there is a separating maximal λ-independent family $\mathcal{S}$ on κ.
(iii) The λ-closed $\mathcal{B}(\mathcal{S}) = \{B_p : p \in Fn(\mathcal{S}, 2; \lambda)\}$ is the base of a 0-dimensional T_2 topology $\tau(\mathcal{S})$ on κ that is irresolvable. So, there is $X(\mathcal{S}) \in \tau(\mathcal{S})^+$ that is OHI.

THEOREM (Kunen)

The existence of a measurable cardinal implies the consistency of $MI(\kappa, \omega_1)$ with $\kappa \leq 2^{\omega_1}$, moreover $MI(\kappa, \omega_1)$ implies CH.

COROLLARY

The existence of a measurable cardinal implies the consistency of having a 0-dimensional T_2-space of size $\leq 2^{\omega_1}$ whose product with any countable irresolvable space is irresolvable.

Part I: irresolvable products 7.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \leq \kappa_1 < \lambda_2 \leq \kappa_2 < \cdots < \lambda_n \leq \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \leq n$ then $\Pi^+(n)$ holds.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \le \kappa_1 < \lambda_2 \le \kappa_2 < \cdots < \lambda_n \le \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \le n$ then $\Pi^+(n)$ holds.

THEOREM

$M(n)$, (resp. $M(\omega)$) implies that there are $\omega < \kappa_1 < \cdots < \kappa_n$ (resp. $\omega < \kappa_1 < \kappa_2 < \cdots$) s.t. $MI(\kappa_i, \kappa_i)$ for each i.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \leq \kappa_1 < \lambda_2 \leq \kappa_2 < \cdots < \lambda_n \leq \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \leq n$ then $\Pi^+(n)$ holds.

THEOREM

$M(n)$, (resp. $M(\omega)$) implies that there are $\omega < \kappa_1 < \cdots < \kappa_n$ (resp. $\omega < \kappa_1 < \kappa_2 < \cdots$) s.t. $MI(\kappa_i, \kappa_i)$ for each i. Consequently,
(i) for $n < \omega$ $M(n)$ implies the consistency of $\Pi^+(n)$;

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \leq \kappa_1 < \lambda_2 \leq \kappa_2 < \cdots < \lambda_n \leq \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \leq n$ then $\Pi^+(n)$ holds.

THEOREM

$M(n)$, (resp. $M(\omega)$) implies that there are $\omega < \kappa_1 < \cdots < \kappa_n$ (resp. $\omega < \kappa_1 < \kappa_2 < \cdots$) s.t. $MI(\kappa_i, \kappa_i)$ for each i. Consequently,
(i) for $n < \omega$ $M(n)$ implies the consistency of $\Pi^+(n)$;
(ii) $M(\omega)$ implies the consistency of having infinitely many 0-dimensional T_2-spaces s.t. the product of any finitely many of them is irresolvable.

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \le \kappa_1 < \lambda_2 \le \kappa_2 < \cdots < \lambda_n \le \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \le n$ then $\Pi^+(n)$ holds.

THEOREM

$M(n)$, (resp. $M(\omega)$) implies that there are $\omega < \kappa_1 < \cdots < \kappa_n$ (resp. $\omega < \kappa_1 < \kappa_2 < \cdots$) s.t. $MI(\kappa_i, \kappa_i)$ for each i. Consequently,
(i) for $n < \omega$ $M(n)$ implies the consistency of $\Pi^+(n)$;
(ii) $M(\omega)$ implies the consistency of having infinitely many 0-dimensional T_2-spaces s.t. the product of any finitely many of them is irresolvable.

PROBLEM. Are $M(n)$ and $\Pi^+(n)$ equiconsistent for any $1 < n \le \omega$?

Part I: irresolvable products 7.

$\Pi^+(n)$ denotes that there are $n+1$ 0-dimensional T_2-spaces whose product is irresolvable. M(n) denotes that there are n measurables.

PROPOSITION

If $\lambda_0 = \omega < \lambda_1 \leq \kappa_1 < \lambda_2 \leq \kappa_2 < \cdots < \lambda_n \leq \kappa_n$ are s.t. $MI(\kappa_i, \lambda_i)$ for each $0 < i \leq n$ then $\Pi^+(n)$ holds.

THEOREM

$M(n)$, (resp. $M(\omega)$) implies that there are $\omega < \kappa_1 < \cdots < \kappa_n$ (resp. $\omega < \kappa_1 < \kappa_2 < \cdots$) s.t. $MI(\kappa_i, \kappa_i)$ for each i. Consequently,
(i) for $n < \omega$ $M(n)$ implies the consistency of $\Pi^+(n)$;
(ii) $M(\omega)$ implies the consistency of having infinitely many 0-dimensional T_2-spaces s.t. the product of any finitely many of them is irresolvable.

PROBLEM. Are $M(n)$ and $\Pi^+(n)$ equiconsistent for any $1 < n \leq \omega$?

Part II: maximal resolvability of products

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

COROLLARY

$MI(\kappa, \omega_1)$ implies the consistency of having a 0-dimensional T_2-space X s.t. $\widehat{c}(X) = \omega_2 < \Delta(X) = |X| \le 2^{\omega_1}$ and $\omega_1 \le add\left(\mathcal{N}(X)\right)$. Then e.g. $X \times \mathbb{Q}$ is ω_2-irresolvable, hence not maximally resolvable.

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

COROLLARY

$MI(\kappa, \omega_1)$ implies the consistency of having a 0-dimensional T_2-space X s.t. $\widehat{c}(X) = \omega_2 < \Delta(X) = |X| \leq 2^{\omega_1}$ and $\omega_1 \leq add\left(\mathcal{N}(X)\right)$. Then e.g. $X \times \mathbb{Q}$ is ω_2-irresolvable, hence not maximally resolvable.

Our OHI X from $MI(\kappa, \omega_1)$ works: $c(X) = \omega_1$ follows from CH;

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa,\kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

COROLLARY

$MI(\kappa,\omega_1)$ implies the consistency of having a 0-dimensional T_2-space X s.t. $\widehat{c}(X) = \omega_2 < \Delta(X) = |X| \leq 2^{\omega_1}$ and $\omega_1 \leq add\left(\mathcal{N}(X)\right)$. Then e.g. $X \times \mathbb{Q}$ is ω_2-irresolvable, hence not maximally resolvable.

Our OHI X from $MI(\kappa,\omega_1)$ works: $c(X) = \omega_1$ follows from CH; $\omega_1 \leq add\left(\mathcal{N}(X)\right)$ follows from the ω_1-closed base.

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

COROLLARY

$MI(\kappa, \omega_1)$ implies the consistency of having a 0-dimensional T_2-space X s.t. $\widehat{c}(X) = \omega_2 < \Delta(X) = |X| \leq 2^{\omega_1}$ and $\omega_1 \leq add\left(\mathcal{N}(X)\right)$. Then e.g. $X \times \mathbb{Q}$ is ω_2-irresolvable, hence not maximally resolvable.

Our OHI X from $MI(\kappa, \omega_1)$ works: $c(X) = \omega_1$ follows from CH; $\omega_1 \leq add\left(\mathcal{N}(X)\right)$ follows from the ω_1-closed base.

Such an example is OHI Baire, so implies $CON(M(1))$.

Part II: maximal resolvability of products

QUESTION. (Ceder and Pearson, 1967) Is the product of a maximally resolvable space with any space maximally resolvable?

Eckertson, 1997: NO from $MI(\kappa, \kappa)$ with $\kappa > \omega$, so his space is "large"!

THEOREM

If X is OHI and $|Y| < \min\left\{\widehat{c}(X), add\left(\mathcal{N}(X)\right)\right\}$ then $X \times Y$ is $\widehat{c}(X)$-irresolvable.

COROLLARY

$MI(\kappa, \omega_1)$ implies the consistency of having a 0-dimensional T_2-space X s.t. $\widehat{c}(X) = \omega_2 < \Delta(X) = |X| \le 2^{\omega_1}$ and $\omega_1 \le add\left(\mathcal{N}(X)\right)$. Then e.g. $X \times \mathbb{Q}$ is ω_2-irresolvable, hence not maximally resolvable.

Our OHI X from $MI(\kappa, \omega_1)$ works: $c(X) = \omega_1$ follows from CH; $\omega_1 \le add\left(\mathcal{N}(X)\right)$ follows from the ω_1-closed base.

Such an example is OHI Baire, so implies $CON(M(1))$.

Part II: maximal resolvability of products 2.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y \rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y\rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$. So, $\bigcap\{E_\alpha : \alpha < \kappa\} = \emptyset$.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y \rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$. So, $\bigcap\{E_\alpha : \alpha < \kappa\} = \emptyset$.

COROLLARY

$M(1)$ implies the consistency of having a monotonically normal space X s.t. $|X| = \Delta(X) = \aleph_\omega$ and $\omega_1 \notin MR(X)$. Thus $X \times \mathbb{Q}$ is ω_1-irresolvable, hence not maximally resolvable.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y\rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$. So, $\bigcap\{E_\alpha : \alpha < \kappa\} = \emptyset$.

COROLLARY

$M(1)$ implies the consistency of having a monotonically normal space X s.t. $|X| = \Delta(X) = \aleph_\omega$ and $\omega_1 \notin MR(X)$. Thus $X \times \mathbb{Q}$ is ω_1-irresolvable, hence not maximally resolvable.

X was constructed by J-Magidor, 2012.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y\rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$. So, $\bigcap\{E_\alpha : \alpha < \kappa\} = \emptyset$.

COROLLARY

$M(1)$ implies the consistency of having a monotonically normal space X s.t. $|X| = \Delta(X) = \aleph_\omega$ and $\omega_1 \notin MR(X)$. Thus $X \times \mathbb{Q}$ is ω_1-irresolvable, hence not maximally resolvable.

X was constructed by J-Magidor, 2012. $\omega_1 \notin MR(X)$ was proved by Soukup, Stanley, 2017.

Part II: maximal resolvability of products 2.

Trivially, $\kappa \in MR(X)$ implies $\kappa \in MR(X \times Y)$.

THEOREM

$|Y| < \kappa = cf(\kappa) \in MR(X \times Y)$ implies $\kappa \in MR(X)$.

PROOF. There is a decreasing κ-sequence $\{D_\alpha : \alpha < \kappa\}$ of dense sets in $X \times Y$ with empty intersection. So, $\{E_\alpha = \pi_X[D_\alpha] : \alpha < \kappa\}$ consists of sets dense in X and is also decreasing. For $x \in X$ there is $\alpha < \kappa$ s.t. $\langle x, y\rangle \notin D_\alpha$ for all $y \in Y$, hence $x \notin E_\alpha$. So, $\bigcap\{E_\alpha : \alpha < \kappa\} = \emptyset$.

COROLLARY

$M(1)$ implies the consistency of having a monotonically normal space X s.t. $|X| = \Delta(X) = \aleph_\omega$ and $\omega_1 \notin MR(X)$. Thus $X \times \mathbb{Q}$ is ω_1-irresolvable, hence not maximally resolvable.

X was constructed by J-Magidor, 2012. $\omega_1 \notin MR(X)$ was proved by Soukup, Stanley, 2017.

A MN and not max. res. X implies $CON(M(1)$ by J-Magidor.

Part II: maximal resolvability of products 3.

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

1. Is the existence of a not maximally resolvable product with a maximally resolvable factor equiconsistent with $M(1)$?

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

1. Is the existence of a not maximally resolvable product with a maximally resolvable factor equiconsistent with $M(1)$?
2. Does it follow from ZFC?

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

1. Is the existence of a not maximally resolvable product with a maximally resolvable factor equiconsistent with $M(1)$?
2. Does it follow from ZFC?
3. What is needed to have a space X and a regular cardinal κ s.t. $\omega < \kappa < \Delta(X)$ and $\kappa \notin MR(X)$?

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

1. Is the existence of a not maximally resolvable product with a maximally resolvable factor equiconsistent with $M(1)$?
2. Does it follow from ZFC?
3. What is needed to have a space X and a regular cardinal κ s.t. $\omega < \kappa < \Delta(X)$ and $\kappa \notin MR(X)$?

I conjecture that if $\omega_1 < |X| = \Delta(X) < \aleph_\omega$ then $\omega_1 \in MR(X)$.

Part II: maximal resolvability of products 3.

All known counterexamples to the Ceder-Pearson problem use $M(1)$.

PROBLEMS.

1. Is the existence of a not maximally resolvable product with a maximally resolvable factor equiconsistent with $M(1)$?
2. Does it follow from ZFC?
3. What is needed to have a space X and a regular cardinal κ s.t. $\omega < \kappa < \Delta(X)$ and $\kappa \notin MR(X)$?

I conjecture that if $\omega_1 < |X| = \Delta(X) < \aleph_\omega$ then $\omega_1 \in MR(X)$.

THANK YOU FOR YOUR ATTENTION!