Free group of Hamel bijections

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Introduction and motivation

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$$(\mathrm{HF} \cup \{\mathrm{id}\}, \circ) \text{ is not a group.}$$

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${}_\xi f_0$	${}_\xi f_1$	....	${}_\xi f_\gamma$	....
... ...	... ...	... ...	... ...	... ...
$f_0 := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_0$	$f_1 := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_1$	....	$f_\gamma := \bigcup_{\alpha<\mathfrak{c}} {}_\alpha f_\gamma$	....

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Indeed, let $\langle x, y \rangle \in \mathbb{R}^2$. Then

$$\langle x, y \rangle = \langle 0, y - f(x) \rangle + \langle x, f(x) \rangle .$$

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Now consider a set W of all the reduced words that can be composed of the generators, i. e. functions of the form

$$h = f_{\gamma_1}^{k_1} \circ \ldots \circ f_{\gamma_m}^{k_m}.$$

where $m \geqslant 1$, $k_i \in \mathbb{Z} \setminus \{0\}$, $\gamma_i < \mathfrak{c}$ and $\gamma_i \neq \gamma_{i+1}$.

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$$W = \{h_\alpha : \alpha < \mathfrak{c}\}.$$

If $W \ni h = f_{\gamma_1}^{k_1} \circ ... \circ f_{\gamma_m}^{k_m}$ then by ${}_\xi h$ we will denote

$${}_\xi f_{\gamma_1}^{k_1} \circ ... \circ_\xi f_{\gamma_m}^{k_m},$$

i. e. the word h_α at the ξ-stage of the construction.

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(I) ${}_\kappa f_\beta$ is a partial function (it has at most one value in every $x \in \mathbb{R}$);

(II) ${}_\kappa f_\beta$ is one-to-one;

(III) ${}_\xi f_\beta \subset_\kappa f_\beta$ for $\xi < \kappa$;

(IV) $|\bigcup_{\gamma<\beta} {}_\kappa f_\gamma| \leqslant |\kappa| + \omega$;

(V) ${}_\kappa h_\beta \in \mathrm{PLIF}$;

(VI) $\langle 0, x_\kappa \rangle \in \mathrm{LIN}_{\mathbb{Q}}({}_{\kappa+1} h_{\alpha_\kappa})$;

(VII) $x_\kappa \in \mathrm{dom}({}_{\kappa+1} f_{\alpha_\kappa})$;

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At the end for every $\beta < \mathfrak{c}$ let

$$f_\beta := \bigcup_{\kappa<\mathfrak{c}} {}_\kappa f_\beta.$$

Why do these conditions suffice?

These conditions assure that for every $\beta < \mathfrak{c}$, $f_\beta \in \mathbb{R}^\mathbb{R}$.

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Indeed, if it was not, some function would have two representations that do not reduce. Composing the function with its inverse would lead to a nontrivial representation of the identity function, a contradiction.

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We will see that this condition will enable us to make the inductive step.

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How do we care about them?

The highlighted conditions are true from the very beginning of our construction. We just need to make sure we don't break any of these.

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On the other hand, conditions (VI)-(VII) are the conditions that we need to satisfy.

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Construction

Assume that for each β (number of generator) ${}_{\xi}f_{\beta}$ are constructed for $\xi < \eta$. If η is a limit ordinal then for each β we let

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If $\langle 0, x_{\kappa} \rangle \in \mathrm{LIN}_{\mathbb{Q}}({}_{\kappa}h_{\alpha_{\kappa}})$, we don't change anything. Let's look at the other case.

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pairs of points and add them to appropiate f_{γ_i}'s in the way that $\langle x, y \rangle, \langle -x, x_\kappa - y \rangle$ are in the extended ${}_{\kappa}h_{\alpha_\kappa}$.

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- (I)-(IV) clearly still hold;
- (V): ${}_{\kappa}h_\beta \in \mathrm{PLIF}$ remains true - we were chosing points that were linearly independent.

PART II and PART III

In these parts we have to satisfy conditions

(VII) $x_\kappa \in \mathrm{dom}({}_{\kappa+1}f_{\alpha_\kappa})$,

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(VIII) $x_\kappa \in \mathrm{rng}({}_{\kappa+1}f_{\alpha_\kappa})$.

The argument showing that it can be done without breaking conditions (I)-(V) is the same - the set of "forbidden" point is not equal to $\mathbb{R}$.
At the end we let ${}_{\kappa+1}f_\beta$ be the extended version of ${}_\kappa f_\beta$ or ${}_{\kappa+1}f_\beta =_\kappa f_\beta$ if it was not changed in parts I-III.

Problems

Problem

Characterize these groups which isomorphic copies can be found within the family of Hamel bijections with identity function included.

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Problem

Does there exists a free group of Hamel bijections with $\mathfrak{c}^+$ generators?

References

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