The refinement relation and its associated cardinal invariants

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Introduction

Given a notion of forcing $\mathbb{P}$, we aim to study the directed set of all maximal antichains in $\mathbb{P}$ from the point of view of Tukey reducibility.

This is a joint work (in progress) with Jörg Brendle.

Relational systems

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- $\mathfrak{b}(\boldsymbol{A}) = \min\{|F| \mid F \subseteq A_- \text{ is unbounded}\}$,
- $\mathfrak{d}(\boldsymbol{A}) = \min\{|D| \mid D \subseteq A_+ \text{ is dominating}\}$.

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Example

Let $\boldsymbol{D} = \langle {}^\omega\omega, {}^\omega\omega, \leq^* \rangle$, where

$$f \leq^* g \iff \text{the set } \{n < \omega \mid g(n) < f(n)\} \text{ is finite.}$$

Then $\mathfrak{b}(\boldsymbol{D}) = \mathfrak{b}$ and $\mathfrak{d}(\boldsymbol{D}) = \mathfrak{d}$.

Tukey reductions

Definition (Tukey [1940])

Let $\boldsymbol{A} = \langle A_-, A_+, A \rangle$ and $\boldsymbol{B} = \langle B_-, B_+, B \rangle$ be relational systems.
A Tukey reduction from $\boldsymbol{A}$ to $\boldsymbol{B}$ consists of two functions

$$\varphi_- \colon A_- \to B_- \quad \text{and} \quad \varphi_+ \colon B_+ \to A_+$$

such that for all $a \in A_-$ and all $b \in B_+$

$$\varphi_-(a) \; B \; b \implies a \; A \; \varphi_+(b).$$

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Write $\boldsymbol{A} \leq_{\mathsf{T}} \boldsymbol{B}$ if there exists a Tukey reduction from $\boldsymbol{A}$ to $\boldsymbol{B}$.
Finally, $\boldsymbol{A} \equiv_{\mathsf{T}} \boldsymbol{B}$ means $\boldsymbol{A} \leq_{\mathsf{T}} \boldsymbol{B}$ and $\boldsymbol{B} \leq_{\mathsf{T}} \boldsymbol{A}$.

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Proposition (Schmidt [1955])

If $\boldsymbol{A} \leq_{\mathrm{T}} \boldsymbol{B}$ then $\mathfrak{d}(\boldsymbol{A}) \leq \mathfrak{d}(\boldsymbol{B})$ and $\mathfrak{b}(\boldsymbol{B}) \leq \mathfrak{b}(\boldsymbol{A})$.

The σ operation

Given a relational system $\boldsymbol{A} = \langle A_-, A_+, A \rangle$, we define

$$\boldsymbol{A}_\sigma = \left\langle A_-, {}^\omega A_+, A_\sigma \right\rangle,$$

where $a \; A_\sigma \; f \iff$ there exists $n < \omega$ such that $a \; A \; f(n)$.

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Remark

- Always $\boldsymbol{A}_\sigma \leq_{\mathrm{T}} \boldsymbol{A}$
- If $\boldsymbol{A} \leq_{\mathrm{T}} \boldsymbol{B}$ then $\boldsymbol{A}_\sigma \leq_{\mathrm{T}} \boldsymbol{B}_\sigma$

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For the purpose of this analysis, we may assume that $\mathbb{P}$ is in fact a complete Boolean algebra.

Almost refinement

Remark

$\mathfrak{b}(\mathbf{Part}(\mathbb{B}))$ is the least cardinal κ such that $\mathbb{B}$ is not κ-distributive. In particular, for many forcing notions of interest (such as Cohen, random...), this bounding number will be countable!

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Lemma

A c.c.c. algebra $\mathbb{B}$ is ${}^{\omega}\omega$-bounding if and only if $\mathfrak{b}(\mathbf{Part}^*(\mathbb{B})) > \aleph_0$.

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If $\mathbb{B}$ is an c.c.c. atomless Boolean algebra, then

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Definition (Horn and Tarski)

A Boolean algebra $\mathbb{B}$ is σ-finite c.c. if there are subsets $S_n \subseteq \mathbb{B}$, for $n < \omega$, such that $\mathbb{B}^+ = \bigcup_{n<\omega} S_n$ and every antichain in S_n is finite.

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Proposition

If $\mathbb{B}$ is a σ-finite c.c. atomless Boolean algebra, then

$$\langle {}^\omega\omega, {}^\omega\omega, \leq^* \rangle \leq_{\mathsf{T}} \mathbf{Part}^*(\mathbb{B}).$$

In particular, it follows that $\mathfrak{b}(\mathbf{Part}^*(\mathbb{B})) \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{d}(\mathbf{Part}(\mathbb{B}))$.

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- Note: If $\mathbb{S}$ is a Suslin algebra, then $\mathfrak{d}(\mathbf{Part}(\mathbb{S})) = \aleph_1$. Such an algebra exists in the Cohen model, where $\mathfrak{d} = 2^{\aleph_0}$.

Definition

Let $\mathcal{B}({}^{\omega}2)$ be the σ-algebra generated by the clopen subsets of the Cantor space ${}^{\omega}2$. Let

$$\mathbb{C}_{\omega} = \mathcal{B}({}^{\omega}2)/\mathcal{M} \quad \text{and} \quad \mathbb{B}_{\omega} = \mathcal{B}({}^{\omega}2)/\mathcal{N}$$

be the quotients modulo the meagre and null ideal, respectively.

Cohen forcing

Theorem (Brendle and P.)

Let nwd *be the ideal of closed nowhere dense subsets of* ${}^{\omega}2$*. Then*

$$\mathbf{Part}(\mathbb{C}_\omega) \equiv_{\mathsf{T}} \langle \mathrm{nwd}, \mathrm{nwd}, \subseteq \rangle.$$

In particular, $\mathfrak{d}(\mathbf{Part}(\mathbb{C}_\omega)) = \mathsf{cof}(\mathcal{M})$ *and* $\mathfrak{b}(\mathbf{Part}(\mathbb{C}_\omega)_\sigma) = \mathsf{add}(\mathcal{M})$*.*

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Corollary

For a notion of forcing $\mathbb{P}$, *the following conditions are equivalent:*

- *if* c *is a Cohen real and* G *is a* $\mathbb{P}$*-generic filter over* V, *then* c *is still a Cohen real in* $V[G]$;
- *for every* $\mathbb{P}$*-name* $\dot{A}$ *of a maximal antichain of* $\mathbb{C}_\omega$ *and every condition* $p \in \mathbb{P}$ *there exists* $q \leq p$ *and a maximal antichain* B *of* $\mathbb{C}_\omega$ *such that* $q \Vdash \dot{A} \preceq \check{B}$;
- $\mathbb{P}$ *preserves the base of the ideal of meagre sets.*

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Idea of the proof.

Since $\mathbb{B}_\omega$ is atomless c.c.c., we have $\langle \mathbb{B}_\omega^+, \mathbb{B}_\omega^+, \geq \rangle_\sigma \leq_{\mathsf{T}} \mathbf{Part}(\mathbb{B}_\omega)_\sigma$ and therefore $\mathfrak{b}(\mathbf{Part}(\mathbb{B}_\omega)_\sigma) \leq \mathfrak{b}\big(\langle \mathbb{B}_\omega^+, \mathbb{B}_\omega^+, \geq \rangle_\sigma\big) = \mathrm{add}(\mathcal{N})$ by a result of Cichoń-Kamburelis-Pawlikowski.

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Conversely, if $\kappa < \mathsf{add}(\mathcal{N})$ then $\mathsf{MA}_\kappa(\mathbb{A})$ holds. Use Amoeba generics to construct a sufficiently "generic" element of $\mathsf{Part}(\mathbb{B}_\omega)$ and conclude that $\kappa < \mathfrak{b}(\mathbf{Part}(\mathbb{B}_\omega)_\sigma)$. □

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Conjecture

We believe the above argument can be dualized to establish that $\mathfrak{d}(\textbf{Part}(\mathbb{B}_\omega)) = \mathsf{cof}(\mathcal{N})$*.*

Questions

- Is $\mathfrak{d}(\mathbf{Part}(\mathbb{B}_\omega)) = \mathrm{cof}(\mathcal{N})$?
- What is the relation between $\mathfrak{d}(\mathbf{Part}(\mathbb{B}))$ and other cardinal invariants of $\mathbb{B}$, in particular the ultrafilter number?

Thank you!