Homogeneous ultrametric structures

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The setting of Fraïssé-theory

Consider an age $\mathscr{C}$ of (model theoretic) structures, i.e.:

- $\mathscr{C}$ consists of countable, finitely generated structures of the same type,
- $\mathscr{C}$ is countable, up to isomorphisms,
- $\mathscr{C}$ has the HP ($\forall \mathbf{A}, \mathbf{B} : \mathbf{B} \in \mathscr{C}, \mathbf{A} \hookrightarrow \mathbf{B} \implies \mathbf{A} \in \mathscr{C}$),
- $\mathscr{C}$ has the JEP ($\forall \mathbf{A}, \mathbf{B} \in \mathscr{C}\, \exists \mathbf{C} \in \mathscr{C} : \mathbf{A} \hookrightarrow \mathbf{C} \hookleftarrow \mathbf{B}$).

Remark

In a model theoretic signature $\Sigma = (\Phi, \mathrm{P}, \mathrm{ar})$ we usually allow:

- *countably many operations and constants ($|\Phi| \leq \aleph_0$),*
- *arbitrarily many relations ($|\mathrm{P}|$ arbitrary).*

Structures from ages

For a structure **U** define

$$\mathrm{Age}(\mathbf{U}) := \{\mathbf{A} \mid \mathbf{A} \hookrightarrow \mathbf{U},\ \mathbf{A} \text{ is finitely generated}\}$$

Proposition (Fraïssé)

$\mathscr{C}$ is an age iff $\mathscr{C} = \mathrm{Age}(\mathbf{U})$, for some countable structure $\mathbf{U}$.

Definition

A structure **V** is called **younger** than $\mathscr{C}$ if $\mathrm{Age}(\mathbf{V}) \subseteq \mathscr{C}$.

$$\sigma\mathscr{C} := \{\mathbf{V} \mid \mathbf{V} \text{ is countable and younger than } \mathscr{C}\}$$

Why the notation $\sigma\mathscr{C}$?

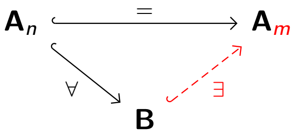
Typically, in Fraïssé-theory structures from $\sigma\mathscr{C}$ are constructed as unions of ω-towers:

Towers

$$\vec{\mathbf{A}} = (\mathbf{A}_i)_{i<\omega} \qquad \forall i<\omega : \mathbf{A}_i \in \mathscr{C} \qquad \forall i<\omega : \mathbf{A}_i \leq \mathbf{A}_{i+1} \qquad \mathbf{A}_\infty := \bigcup_{i<\omega} \mathbf{A}_i.$$

E.g., for an ω-tower $\vec{\mathbf{A}} = (\mathbf{A}_i)_{i<\omega}$ over $\mathscr{C}$ we have:

$\mathbf{A}_\infty$ is universal and homogeneous $\iff$ $\vec{\mathbf{A}}$ has the absorption property:



Fact

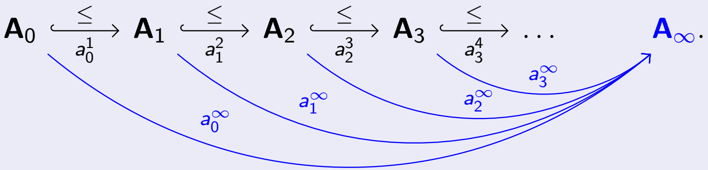
$\sigma\mathscr{C}$ consists exactly of the class of unions of ω-towers over $\mathscr{C}$.

$\sigma\mathscr{C}$ comes from "Sum" of ω-towers of structures from $\mathscr{C}$.

ω-towers as chains

Towers are special ω-chains and their unions are colimits

- Consider $\boldsymbol{\omega} = (\omega, \leq)$ as a category;
- Consider the category $(\mathscr{C}, \hookrightarrow)$;
- Each ω-tower $\overrightarrow{\mathbf{A}} = (\mathbf{A}_i)_{i<\omega}$ defines a functor from $\boldsymbol{\omega}$ to $(\mathscr{C}, \hookrightarrow)$ (an ω-chain);
- the union $\mathbf{A}_\infty$ is a colimit of this functor:



Dualizing ω-towers

Given a class $\mathscr{C}$ of structures of the same type.

Dual towers

- Consider the category $(\mathscr{C}, \twoheadrightarrow)$;
- A dual ω-tower is a functor from $\boldsymbol{\omega}^{\mathrm{op}}$ to $(\mathscr{C}, \twoheadrightarrow)$ (an ω-cochain);
- A dual tower $\overleftarrow{\mathbf{A}}$ over $\mathscr{C}$ is determined by

$$\mathbf{A}_0 \xleftarrow{\alpha_0^1} \mathbf{A}_1 \xleftarrow{\alpha_1^2} \mathbf{A}_2 \xleftarrow{\alpha_2^3} \mathbf{A}_3 \xleftarrow{\alpha_3^4} \dots .$$

Limits of dual towers

- The limiting structure $\mathbf{A}_\infty = \varprojlim \overleftarrow{\mathbf{A}}$ consists of all those tuples $(x_i)_{i<\omega}$ with
  1. $\forall i < \omega : x_i \in A_i$,
  2. $\forall i < \omega : x_i = \alpha_i^{i+1}(x_{i+1})$;
- Operations and relations are defined coordinate-wise;
- The limiting cone is given by the projections:

$$\mathbf{A}_0 \xleftarrow{\alpha_0^1} \mathbf{A}_1 \xleftarrow{\alpha_1^2} \mathbf{A}_2 \xleftarrow{\alpha_2^3} \mathbf{A}_3 \xleftarrow{\alpha_3^4} \dots \qquad \mathbf{A}_\infty .$$

$$\alpha_0^\infty \colon \mathbf{A}_\infty \twoheadrightarrow \mathbf{A}_0,\quad \alpha_1^\infty \colon \mathbf{A}_\infty \twoheadrightarrow \mathbf{A}_1,\quad \alpha_2^\infty \colon \mathbf{A}_\infty \twoheadrightarrow \mathbf{A}_2,\quad \alpha_3^\infty \colon \mathbf{A}_\infty \twoheadrightarrow \mathbf{A}_3$$

Are there interesting limits of dual towers of structures?

Answer 1

T. Irwin and S. Solecki. Projective Fraïssé limits and the pseudo-arc. *Trans. Amer. Math. Soc.*, 358(7):3077–3096, 2006.

Objection

But they describe structures that are dually universal and dually homogeneous!

We would like to get structures that are **universal** and **homogeneous** in a classical sense.

Are there interesting limits of dual towers of structures? (cont.)

Towards Answer 2

Step 1: Fix a class of **large structures** together with a **notion of embeddings**;

Step 2: Fix a notion of **small structures** .

The proper decision in these two steps leads to a Fraïssé-type result.

A putative universal homogeneous structure in the class of large structures

- should embed every other large structures,
- should be homogeneous with respect to small substructures, i.e., every isomorphism between small substructures should extend to an automorphism.

Structures and embeddings

- Fix $\mathscr{D}$ (usually of shape $\sigma\mathscr{C}$ for an age $\mathscr{C}$);
- Consider the category $(\mathscr{D}, \twoheadrightarrow)$;
- Take as (large) structures all structures of the shape

$$\mathbf{A}_\infty = \varprojlim \overleftarrow{\mathbf{A}}, \quad \text{where} \quad \overleftarrow{\mathbf{A}} : \omega^{\mathrm{op}} \to (\mathscr{D}, \twoheadrightarrow)$$

- As embeddings take model theoretic embeddings.

Problem: These are too many embeddings!

- $\mathbf{A}_\infty$ carries a natural ultrametric:
- For $\mathbf{a} = (a_i)_{i<\omega}$, $\mathbf{b} = (b_i)_{i<\omega}$ define

$$d(\mathbf{a}, \mathbf{b}) := \begin{cases} 0 & \mathbf{a} = \mathbf{b}, \\ 2^{-D(\mathbf{a},\mathbf{b})} & \text{else}, \end{cases} \quad \text{where } D(\mathbf{a}, \mathbf{b}) = \min\{i < \omega \mid a_i \neq b_i\}.$$

Embeddings that do not preserve this ultrametric are problematic.

Solution (attempt)

Add the canonical ultrametric to the structures and consider only isometric embeddings.

Does adding an ultrametric already solve our problem?

Good news

For purely algebraic structures this is indeed good enough.

Bad news

As soon as we add relations to our structures, we get into trouble.

Problem

For any $(\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(n-1)}) \in A_\infty^n$ we have

$$(\mathbf{a}^{(0)}, \dots, \mathbf{a}^{(n-1)}) \notin \varrho^{\mathbf{A}_\infty} \iff \exists i < \omega : (a_i^{(0)}, \dots, a_i^{(n-1)}) \notin \varrho^{\mathbf{A}_i}.$$

Current isometric embeddings do not take into account, for which i this happens.

Solution

Add this information somehow to the structure and reduce the class of embeddings accordingly.

Replacing two-valued by many-valued predicates

- Let $\mathbf{A}_\infty = \varprojlim \overleftarrow{\mathbf{A}}$, for $\overleftarrow{\mathbf{A}} : \omega^{\mathrm{op}} \to (\mathscr{D}, \twoheadrightarrow)$;
- For every relational symbol ϱ (say, of arity n), we define a predicate

$$\underline{\varrho}^{\mathbf{A}_\infty} : A_\infty^n \to [0,1] \text{ through}$$

$$\underline{\varrho}^{\mathbf{A}_\infty}(\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(n-1)}) := \begin{cases} 0, & (\mathbf{x}^{(0)}, \dots, \mathbf{x}^{(n-1)}) \in \varrho^{\mathbf{A}_\infty} \\ 2^{-\min\{i<\omega \mid (x_i^{(0)}, \dots, x_i^{(n-1)}) \notin \varrho^{\mathbf{A}_i}\}}, & \text{else.} \end{cases}$$

Proposition

$\mathcal{A}_\infty = (A_\infty, d, (f^{\mathbf{A}_\infty})_{f\in\Phi}, (\underline{\varrho}^{\mathbf{A}_\infty})_{\varrho\in\mathrm{P}})$ *is a metric structure in the sense of*

I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov. Model theory for metric structures. In Model theory with applications to algebra and analysis. Vol. 2, *volume 350 of* London Math. Soc. Lecture Note Ser., *pages 315–427. Cambridge Univ. Press, Cambridge, 2008.*

In particular, all operations and all predicates are 1-Lipschitz.

Large structures

$$\mathcal{A}_\infty = \varprojlim \overleftarrow{\mathbf{A}}, \qquad \pi\mathscr{D} := \{\varprojlim \overleftarrow{\mathbf{A}} \mid \overleftarrow{\mathbf{A}} : \omega^{\mathrm{op}} \to (\mathscr{D}, \twoheadrightarrow)\}$$

Ultrametric structures and metric embeddings

Definition

Let $\mathcal{A} = (A, \delta_{\mathcal{A}}, (f^{\mathcal{A}})_{f \in \Phi}, (\varrho^{\mathcal{A}})_{\varrho \in \mathrm{P}})$ be a metric Σ-structure. Then $\mathcal{A}$ is called an **ultrametric Σ-structure** if

1. $(A, \delta_{\mathcal{A}})$ is an ultrametric space of diameter ≤ 1,
2. $f^{\mathcal{A}}$ is 1-Lipschitz, for each $f \in \Phi$,
3. $\varrho^{\mathcal{A}} \colon A^{\mathrm{ar}(\varrho)} \to [0,1]$ is 1-Lipschitz, for each $\varrho \in \mathrm{P}$.

Definition

For metric structures $\mathcal{A}$ and $\mathcal{B}$ an injection $\iota \colon A \hookrightarrow B$ is called **metric embedding** if

1. ι is an isometry,
2. ι preserves all operations (in the usual sense),
3. ι preserves all (many-valued) predicates.

Possible notions of smallness in $\pi\mathscr{D}$

- Given an age $\mathscr{C}$,
- Consider $\mathscr{D} = \sigma\mathscr{C}$.

An ultrametric structure $\mathcal{A} \in \pi\mathscr{D}$ may be

1. finite,
2. finitely generated,
3. compact,
4. compactly generated.

We choose a notion of smallness that, in general, subsumes all of the former:

Definition

$\mathcal{A} \in \pi\mathscr{D}$ is called **profinitely generated** $:\iff \mathcal{A} \in \pi\mathscr{C}$

For purely relational structures we have

profinitely generated $\iff$ compactly generated $\iff$ compact

Universal homogeneous ultrametric structures

- Given an age $\mathscr{C}$.
- An ultrametric structure $\mathcal{U} \in \pi\sigma\mathscr{C}$ is called

  universal if every $\mathcal{V} \in \pi\sigma\mathscr{C}$ metrically embeds into $\mathcal{U}$,

  homogeneous every metric isomorphism between profinitely generated metric substructures of $\mathcal{U}$ extends to a metric automorphism of $\mathcal{U}$.

Theorem

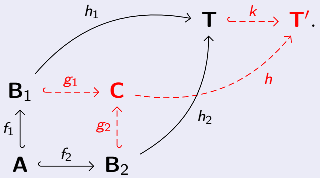
$\pi\sigma\mathscr{C}$ contains a universal homogeneous ultrametric structure **if and only if**

- *$\mathscr{C}$ has the* AP,
- *$\mathscr{C}$ has the* AEP *(Amalgamated Extension Property).*

Moreover, any two universal homogeneous ultrametric structures in $\pi\sigma\mathscr{C}$ are metrically isomorphic.

Note

$\mathscr{C}$ has the **amalgamated extension property** (AEP) if . . .



Concerning the AEP

- It is easier to find examples for ages with the AEP than counter examples.
- If $\mathscr{C}$ has the free AP, then it has the AEP (e.g., graphs, K_n-free graphs,...).
- If $\mathscr{C}$ has pushouts (in a suitable sense) then it has the AEP (finite posets, finite rational metric spaces, finite semilattices, finite distributive lattices, finite Boolean algebras...).
- For some $\mathscr{C}$ the AEP may be proved ad hoc (e.g., finite chains).

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2. Slowly go closer . . .
3.
    - Every vertex starts to look like a copy of R,
    - Every edge starts to look like a random bipartite graph,
    - The graph altogether looks again like a copy of R

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4. Proceed with step 2.