Adding a club of former regulars to an inaccessible cardinal.

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SETTOP 2022
23rd August, 2022

Joint work with Moti Gitik.

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- More forcings: Tree Prikry forcing, supercompact Prikry forcing, diagonal Prikry forcing, Prikry-type forcings with interleaved forcings, ...

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- (Foreman, Woodin) GCH fails everywhere.
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- (Benhamou, Garti, Poveda) failure of Galvin's property at every successor of a singular cardinal.

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V-Regular cardinals outside the club may be singularized at the first stage.

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- $\mathbb{P}_\kappa = \cup_{\alpha<\kappa} P_\alpha$, where $p \leq q$ if for some α, $p \restriction P_\alpha \leq_{P_\alpha} q$.
- The forcing P_α is a Prikry-type forcing, and nice. However, $\mathbb{P}_\kappa$ is no longer of Prikry-type.

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- The *one-step extension* using $\gamma \in A$ is $\langle \gamma \rangle^\frown \langle F(\gamma), H(\gamma) \rangle^\frown \langle \alpha, A \setminus \gamma + 1, F \restriction (A \setminus \gamma + 1), H \restriction (A \setminus \gamma + 1) \rangle$.

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- For $\circ(\alpha) > 1$, the process is similar, except that when we perform a one-step extension, there is some reflection.
- Define $\mathbb{P}_\kappa = \cup_{\alpha<\kappa} P_\alpha$ and the ordering is as before.

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Lemma (Key lemma)

For $\nu < \kappa$ and $f : \nu \to ON$, $f \in V[G]$, there is $\alpha < \kappa$ such that $f \in V[G \restriction P_\alpha]$.

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- By elementarity, find $q \in M[G]$. Say $q \in P_\xi / G$, $\gamma \leq \xi < M \cap \kappa = \alpha$. Let $\dot{q}$ and $F(\gamma)$ be P_γ-names for such q and ξ. Set $H(\gamma) = \dot{q}$.

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- For any $\beta < \circ(\alpha)$: each γ with $\circ(\gamma) = \beta$, find an associated $\dot{q}$ deciding $\dot{f}(\beta)$. Then forcing with $\langle \alpha, A, F, H \rangle$ already decides $\dot{f}$, so $\dot{f}[G] \in V[G \restriction P_\alpha]$.

Thank you!