Alternatives to Halpern and Läuchli's theorem

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Trees

- A tree is partial ordering with the smallest element, where for every *a* from the domain the set of all elements smaller than *a* is well ordered.
- Branches are maximal chains. Here we will consider trees where each node has at least one, but finitely many immediate successors, and where each node has finitely many predecessors.
- Also, we limit our considerations to the case where each branch has infinitely many nodes that have more than one immediate successor.

Nodes

- The height of a node is the number of elements smaller than that node.
- The m-th layer in the tree T, denoted by $T(m)$, is the set of all its nodes of height m.

Product of the trees

Let $T_1, \ldots, T_d$ be a sequence of trees as ordered structures. The domain of their product is the set

$$\bigcup_{m=0}^{\infty} (T_1(m) \times \cdots \times T_d(m)).$$

We define $(a_1, \ldots, a_d) < (b_1, \ldots, b_d)$ as $a_1 < b_1, \ldots, a_d < b_d$.

Halpern and Läuchli's theorem

Let f maps the set

$$\bigcup_{m=0}^{\infty} (T_1(m) \times \cdots \times T_d(m)).$$

into finite set (coloring by finite many colors). Then there are $m, n, t_1, \ldots, t_d, D_1, \ldots, D_d$ such that it holds

- $m < n$, $t_1 \in T_1(m), \ldots, t_d \in T_d(m)$, $D_1 \subseteq T_1(n), \ldots, D_d \subseteq T_d(n)$,
- For every i and every immediate successor s of t_i in T_i there is a node $w \in D_i$ which is above the s in T_i.
- f is constant on $D_1 \times \cdots \times D_d$ (the same color).

For given trees and number of colors, there is n' such that choice is possible to satisfy $n \leqslant n'$. Here n' does not depend on coloring.

Cohen's symmetric model

- Let M be a model and G is generic filter over M for the poset of all finite functions from ω^2 to $\{0,1\}$ and $f = \bigcup G$.
- Let $c_i(n) = f(i,n)$ and $C = \{c_i \mid i \in \omega\}$.
- The Cohen's symmetric model over M is the model $N = (HOD(M \cup C \cup \{C\}))^{M[G]}$.
- It holds $M \subseteq N \subseteq M[G]$.

AC in Cohen Symmetric model

- $N \models \mathrm{ZF} + \neg\mathrm{DC}$.
- $N \models \mathrm{BPI}$.
- Halpern and Läuchli's theorem was formulated for the proof that $N \models \mathrm{BPI}$.

Neighborhoods

- For each string p we define $[p]$ as set of all $c \in C$ such that $p \subseteq c$ holds.
- Neighborhoods are sets of the form $[p]$ for some string p.
- Neighborhood of $c \in C$ is neighborhood $[p]$ such that $c \in [p]$.

The continuity lemma

Let $a_1, \ldots, a_m \in M$, $r_1, \ldots, r_n$ are distinct members of C and $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n, z)$ is any formula. Then there are mutually disjoint neighborhoods $[p_1], \ldots, [p_n]$ of $r_1, \ldots, r_n$ respectively such that for every $r'_1 \in [p_1], \ldots, r'_n \in [p_n]$ it holds

$$N \models \varphi(a_1, \ldots, a_m, r'_1, \ldots, r'_n, C)$$

iff

$$N \models \varphi(a_1, \ldots, a_m, r_1, \ldots, r_n, C).$$

The smallest set of parameters

- For every $a \in N$ there is the smallest subset $F_1(a)$ of C such that
$$a \in (HOD(M \cup F_1(a) \cup \{C\}))$$
holds.
- The set $F_1(a)$ is finite.
- F_1 is the class of $M[G]$ with parameters from $M \cup \{C\}$.

Finite range property and DC

- For every $s \in N$ such that $s : \omega \longrightarrow C$ it holds $|s[\omega]| < \aleph_0$.
- Let us define $f : \mathbb{R}^N \longrightarrow \{0, 1\}$ as follows: $f(x) = 1$ iff $x \in C$.
- $f \in N$ and because of density of C in $(0, 1)$ the function f is nowhere continuous in $(0, 1)^N$, but for every $q \in (0, 1) \cap \mathbb{Q}$ and for every $x \in N$ such that $x : \omega \longrightarrow \mathbb{R}^N$ it holds

$$\lim_{n\to\infty} x(n) = q \Rightarrow \lim_{n\to\infty} f(x(n)) = 0 = f(q).$$

- $N \not\models \mathrm{DC}$.

Small choice violation lemma

Fix we some finite $A \subseteq C$ and some $\alpha \in ORD \cap N$. Then, in the model N there is well ordering of the set of all $a \in N$ such that

$$N \models (a \in V_\alpha \wedge F_1(a) \subseteq A).$$

Alternative method for Halpern and Läuchli's theorem

- The Halpern and Läuchli's theorem has also been used to prove other theorems.
- We give a method that can be used instead of the Halpern and Läuchli's theorem to prove other theorems in whose proofs the Halpern and Läuchli's theorem is applicable.
- This method can (not) be used to prove the Halpern and Läuchli's theorem itself.

Alternative method for Halpern and Läuchli's theorem

- The method is based on the fact that in the Cohen symmetric model there is a non-principal ultrafilter over ω.
- This fact is proved by applying the Halpern and Läuchli's theorem, so such a derivation is not an alternative proof of the Halpern and Läuchli's theorem.
- It shows that this method has the power of the Halpern and Läuchli's theorem.

Encoding of branches

Let T be a tree and b be one of its branches. We assign f, g and h sequences to branch b as follows: Let $f(n)$ be the node of branch b at height n, $g(n)$ the number of children of node $f(n)$ and $h(n)$ the sequence number of node $f(n+1)$ among the children of node $f(n)$. Therefore, $1 \leqslant h(n) \leqslant g(n)$ holds.

For example, if the array g starts with 1,5,3,... and the array h starts with 1,3,2,... then node $f(1)$ is the only child of node $f(0)$, node $f(1)$ has five children of which $f(2)$ is the third, and node $f(2)$ has three children of which $f(3)$ is the second.

Encoding of nodes

Let $u(n)$ be a sequence of $h(n) - 1$ ones if $h(n) = g(n)$, and a sequence of $h(n) - 1$ ones followed by one zero if $h(n) < g(n)$. To the branch b we will associate a string obtained by concatenating strings of the form $u(n)$, where $n \in \omega$. This defines one natural bijection between all branches of the tree T and all infinite sequences of zeros and ones.

To the node $f(n)$ for $n > 0$ we will associate the sequence obtained by concatenating the sequences $u(0), \ldots, u(n-1)$. We will associate the node $f(0)$ with an empty string.

Cohen branches

Branches corresponding to strings from the set C will be called Cohen's branches.

Due to the density of the set C, infinitely many Cohen branches pass through each node of the tree.

Continuity Lemma for Cohen Branches

Let $\varphi(k_1, \ldots, k_m, i_1, \ldots, i_n, z, u_1, \ldots, u_d)$ be any formula and $a_1, \ldots, a_m \in M$. Let $r_1, \ldots, r_n$ be the mutually distinct elements of the set C and $b_1, \ldots, b_d$ be the Cohen branches of the trees $T_1, \ldots, T_d$ respectively corresponding to the mutually distinct elements of the set $C \setminus \{r_1, \ldots, r_n\}$.

Continuity Lemma for Cohen Branches

Then there are nodes $t_1, \ldots, t_d$ on branches $b_1, \ldots, b_d$, as well as disjoint neighborhoods $[s_1], \ldots, [s_n]$ of elements $r_1, \ldots, r_n$ so that for any Cohen branches $b'_1, \ldots, b'_d$ of trees $T_1, \ldots, T_d$ containing the specified nodes and any $r'_1, \ldots, r'_n$ from the specified neighborhoods are considered to be branches of $b'_1, \ldots, b'_n$ correspond to mutually different elements of the set $C \setminus ([s_1] \cup \cdots \cup [s_n])$, as well as

$$N \models \varphi(a_1, \ldots, a_m, r'_1, \ldots, r'_n, C, b'_1, \ldots, b'_d)$$

iff

$$N \models \varphi(a_1, \ldots, a_m, r_1, \ldots, r_n, C, b_1, \ldots, b_d).$$

Let F be a non-principal ultrafilter over ω in N and let $a_1, \ldots, a_m \in M$ and $r_1, \ldots, r_n$ be mutually distinct elements of the set C such that F definable in $M[G]$ with parameters $a_1, \ldots, a_m, r_1, \ldots, r_n, C$.
Let $b_1, \ldots, b_d$ be any branches of the tree $T_1, \ldots, T_d$, corresponding to mutually different elements of the set $C \setminus \{r_1, \ldots, r_d\}$.
If we denote by $b(n)$ the node of the branch b at the height n, then we can associate the color c with the set

$$A_c = \{n \in \omega \,|\, f(b_1(n), \ldots, b_d(n)) = c\},$$

then $\bigcup_c A_c = \omega$ holds, so we can choose the color c so that $A_c \in F$ holds.

Derivation of HLT from validity of BPI in N

So it's valid

$$N \models \{n \in \omega \mid f(b_1(n), \ldots, b_d(n)) = c\} \in F,$$

so there are nodes $t_1, \ldots, t_d$ on branches $b_1, \ldots, b_d$ such that for any Cohen branches $b'_1, \ldots, b'_d$ of trees $T_1, \ldots, T_d$ containing nodes $t_1, \ldots, t_d$ and which correspond to mutually different elements of the set $C \setminus \{r_1, \ldots, r_n\}$ holds

$$N \models \{n \in \omega \mid f(b'_1(n), \ldots, b'_d(n)) = c\} \in F.$$

Without loss of generality, we can assume that all of the nodes $t_1, \ldots, t_d$ are at the same height m.

For each i, let $w^i_1, \ldots, w^i_{k_i}$ be all mutually distinct children of node t_i in tree T_i. For each i and each j, let b^i_j be the Cohen branch of the tree T_i containing the node w^i_j. Let's choose them so that they correspond to mutually different elements of the set $C \setminus \{r_1, \ldots, r_n\}$.

Derivation of HLT from validity of BPI in N

Each sequence $l_1, \dots, l_d$ such that for each i holds $1 \leqslant l_i \leqslant k_i$ corresponds to a set

$$S(l_1, \dots, l_d) = \{n \in \omega : f(b^1_{l_1}(n), \dots, b^d_{l_d}(n)) = c\},$$

which at the same time belongs to the filter F. Let S be the intersection of all sets of that form. It will also belong to the filter F, so there exists some $n \in F$ such that $n > m$. Let

$$D_i = \{b^i_1(n), \dots, b^i_{k_i}(n)\}.$$

Then above each child node t_i there is an element of the set D_i and the function f is constant on the set $D_1 \times \cdots \times D_d$.

What is this for?

- Here, the fact that model N satisfies BPI is used, which is proved using the Halpern and Läuchli's theorem, so this is not an alternative proof of the Halpern and Läuchli's theorem.
- However, the Halpern and Läuchli's theorem is also used in proofs of other statements.
- In cases where the Halpern and Läuchli's theorem is applicable, we can use this method as an alternative to the Halpern and Läuchli's theorem.

Research by Andy Zucker

- Andy Zucker tried to find a reformulation of the Halpern and Läuchli's theorem in the language of branches instead of the language of nodes.
- Such reformulation is topological.
- All trees are topologically equivalent to the Cantor set.

Tree topology

- We will denote the set of all branches of the tree T by $[T]$.
- For the node t of the tree T, we will denote by N_t the set of all branches of the tree T that contain the node t.
- On the set $[T]$, we define a topology such that the set of all sets of the form N_t, where t is a node of the tree T, is the set of base open sets.
- In that topology, each tree is equivalent to a Cantor set, which we will denote by K. Also, it holds

$$[T_1 \otimes \cdots \otimes T_d] = [T_1] \times \cdots \times [T_d].$$

Let $T_1, \ldots, T_d$ be any trees and let $t_1, \ldots, t_d$ be some nodes of those trees. A set $S \subseteq [T_1] \times \cdots \times [T_d]$ is said to be one $(t_1, \ldots, t_d)$-DDF if the following holds:

- Set $\{x_1 \in N_{t_1} \mid (\exists x_2, \ldots, x_d)(x_1, \ldots, x_d) \in S\}$ is dense in N_{t_1} in tree T_1.
- For each $k < d$ and any $(x_1^1, \ldots, x_d^1), \ldots, (x_1^m, \ldots, x_d^m) \in S$, the set $\{y_{k+1} \mid (\exists y_{k+2}, \ldots, y_d) \bigwedge_{i=1}^m (x_1^i, \ldots, x_k^i, y_{k+1}, \ldots, y_d) \in S\}$ is dense in $N_{t_{k+1}}$ in tree T_{k+1}.

A set S is said to be somewhere-DDF if there are nodes $t_1, \ldots, t_d$ of the tree $T_1, \ldots, T_d$ such that S is a DDF set over $(t_1, \ldots, t_d)$. Obviously, these notions are topological.

- We will denote the following statement by Z_d: For any partition of the set $[T_1] \times \cdots \times [T_d]$ into finitely many parts, it holds that at least one of the parts has a subset that is somewhere-DDF.
- In other words, the set of all subsets of the set $[T_1] \times \cdots \times [T_d]$ that do not have a subset that is somewhere-DDF is a proper ideal.
- Also, one of the equivalent formulations is that there exists an ultrafilter over the set $[T_1] \times \cdots \times [T_d]$ whose every element contains a subset that is somewhere-DDF.

We will denote the statement $(\forall d) Z_d$ by Z.

Zucker's results

- Zucker proves that Z_1 and Z_2 hold.
- Zucker proves that $(\forall d)(Z_d \Rightarrow \mathrm{HLT}_d)$ holds, where HLT_d is a statement of the Halpern and Läuchli's theorem for the dimension d. The axioms of ZF+BPI are sufficient for the proof.
- Therefore, Zucker found a new proof for HLT_2.
- Zucker proves that $\mathrm{ZFC} + \mathrm{CH} \vdash \neg \mathrm{Z}_d$ holds for $d \geqslant 3$.

How to find models for Z?

- Zucker constructed a counterexample using the axioms ZF+AC+CH.
- If we limit our considerations to models for ZF, then the model for Z must either not satisfy AC or not satisfy CH.
- Cohen's symmetric model does not satisfy AC.
- The model for ZFC+Z must not satisfy CH. Statement Z is related to the Halpern and Läuchli's theorem, which Harrington proved by forcing using a model in which CH does not hold.

Z in Cohen's symmetric model

- In Cohen's symmetric model, the following generalization of Zuker's principle holds: Every partition of the set K^d into finitely many parts contains a part that has a subset that is the Cartesian product of d somewhere dense sets.
- The proof is simple and is performed by applying the continuity lemma. It is also possible to use the mentioned method with Cohen's branches.
- We will denote this generalization of Zuker's principle by Z'.
- Proof of the consistency of Z with ZFC is more complex and requires certain preparations.

Some cardinal equation

- Fix we an infinite cardinal λ. What about class of all infinite cardinals κ such that $\kappa^\lambda = \kappa$ holds?
- For every $\mu \geqslant \lambda$, $\kappa = 2^\mu$ is a solution.
- If $\kappa = \kappa_0$ is solution, then $\kappa = \kappa_0^+$ is also solution.
- The class of all regular solutions is the proper class.

The Δ-system lemma

Let λ be any infinite cardinal and let κ and θ be regular uncountable cardinals with

$$\kappa^{<\lambda} = \kappa \geqslant \lambda, \quad \theta = \kappa^+.$$

Let X, Y, and D be any sets such that $|Y| \leqslant \kappa$ and $|D| = \theta$. Let p be a function that assigns to each $x \in D$ some partial function from X in Y of cardinality less than λ. Then there exists a set $D' \subseteq D$ equipotent to D, such that the set $p[D']$ is a Δ-system. If $D = \theta$, the set D' can be chosen to be stationary.

The constancy lemma

Let Y be any set and let $X_1, \ldots, X_d$ be sets whose cardinal numbers are uncountable regular cardinals with $|X_1| > |Y|$ and $|X_{i+1}| > 2^{|X_i|}$ for all $i < d$. Then, for any $f : X_1 \times \cdots \times X_d \longrightarrow Y$ there are sets $X_1' \subseteq X_1, \ldots, X_d' \subseteq X_d$ with $|X_1'| = |X_1|, \ldots, |X_d'| \subseteq |X_d|$ and such that the function f is constant on the set $X_1' \times \cdots \times X_d'$.
If $X_1, \ldots, X_d$ are cardinals, then X_1

Assume that κ and λ are infinite cardinals with $\kappa^{<\lambda} = \kappa \geqslant \lambda$. Let X and Y be any sets with $|X|, |Y| \leqslant \kappa$. Let P denote the set of all partial functions from X in Y of cardinality less than λ. Let $X_1, \dots, X_d$ be arbitrary sets whose cardinal numbers are uncountable regular cardinals with

1. $|X_1| = \kappa^+$,
2. for every $i < d$ there is some $\mu_i \geqslant 2^{|X_i|}$ such that $|X_{i+1}| = \mu_i^+$ and $\mu_i^{|X_i|} = \mu_i$ hold.

Then for each $p : X_1 \times \cdots \times X_d \longrightarrow P$ there are sets $X_1' \subseteq X_1, \dots, X_d' \subseteq X_d$ such that

$$|X_1'| = |X_1|, \dots, |X_d'| = |X_d|$$

and such that the union $\bigcup p[X_1' \times \cdots \times X_d']$ is a function.

Let $X_1, \dots, X_d, D$ and Y be the sets, p the mapping of the set $X_1 \times \cdots \times X_d$ into the set of partial functions from D to Y and $r_1, \dots, r_d$ partial functions from D to Y. We will say that $(p, X_1, \dots, X_d, r_1, \dots, r_d)$ is a Δ^d-system if the following holds true:

Δ^d-system

1. $r_1 \subseteq \cdots \subseteq r_d$,
2. for each $k < d$ and all $x_1 \in X_1, \ldots, x_d \in X_d$ the set

$$\{p(x_1, \ldots, x_d) \cap r_{k+1} \,|\, x_1 \in X_1, \ldots, x_d \in X_d\}$$

   forms a Δ-system with the root r_k,
3. for each $k < d$ the value of $p(x_1, \ldots, x_d) \cap r_{k+1}$ depends only on $x_1, \ldots, x_k$,
4. for each q there are sets $F_1, \ldots, F_d$ of cardinality not greater than $|q|$ such that the following holds:
   1. The statement $\mathrm{dom}(p(\bar{x})) \cap q \subseteq \mathrm{dom}(r_1)$ holds for every $\vec{x} \in (X_1 \setminus F_1) \times \cdots \times (X_d \setminus F_d)$.
   2. The statement $\mathrm{dom}(p(\bar{x})) \cap q \subseteq \mathrm{dom}(r_{k+1})$ holds for each $k < d$ and any

$$\vec{x} \in X_1 \times \cdots \times X_k \times (X_{k+1} \setminus F_{k+1}) \times \cdots \times (X_d \setminus F_d).$$

A multidimensional variant of the Δ-system lemma

Let κ and λ be any infinite cardinals such that $\kappa^{<\lambda} = \kappa \geqslant \lambda$ and let $\kappa_1, \ldots, \kappa_d$ be cardinals such that that $\kappa_1 = \kappa^+$ and that for every $i < d$ there is some $\mu_i \geqslant 2^{\kappa_i}$ such that $\kappa_{i+1} = \mu_i^+$ and $\mu_i^{\kappa_i} = \mu_i$ hold. Let $X_1, \ldots, X_d$ be sets of cardinality $\kappa_1, \ldots, \kappa_d$ respectively, let $D = X_1 \cup \cdots \cup X_d$ and let Y be a nonempty set such that $|Y| \leqslant \kappa$. Let $p : X_1 \times \cdots \times X_d \longrightarrow \mathrm{Fn}(D, Y, \lambda)$ be such that

$$(\forall x_1 \in X_1, \ldots, x_d \in X_d)\{x_1, \ldots, x_d\} \subseteq \mathrm{dom}(p(x_1, \ldots, x_d))$$

holds. Then there exist $X'_1, \ldots, X'_d$ and $r_1, \ldots, r_d$ such that the following holds:

1. $(p, X'_1, \ldots, X'_d, r_1, \ldots, r_d)$ is a Δ^d-system,
2. $\bigwedge_{i=1}^{d}(X'_i \subseteq X_i \wedge |X'_i| = |X_i|)$,
3. $|r_1| < \lambda$ and $\bigwedge_{i=2}^{d} |r_i| \leqslant \kappa_{d-1}$.

The idea of proof

- Let's illustrate the idea of the proof on the case where $d = 2$.
- For a fixed $\beta \in X_2$ we have a system of functions by $\alpha \in X_1$, as well as a corresponding Δ-subsystem.
- If the cardinal number of the set X_2 is regular and greater than the cardinal number of such Δ-systems, there will be many values for β that give the same Δ-system.
- The proof is performed by induction on d. For $d = 1$ the statement reduces to the standard Δ-system lemma.

A generalization of Zucker's principle

- By $Z''_d(\kappa)$, where κ is an infinite cardinal, we denote the statement that every partition of the set K^d into less than κ parts has a part that has a subset that is somewhere DDF.
- With $Z''(\kappa)$ we will denote the statement $(\forall d)Z''_d(\kappa)$.

Consistency of Z_d with ZFC

Let M be a countable transitive model. Let $\kappa_1, \ldots, \kappa_d$ be uncountable regular cardinals in M such that for every $i < d$ there exists some $\mu_i \geqslant 2^{\kappa_i}$ such that $\mu_i^{\kappa_i} = \mu_i$ and $\kappa_{i+1} = \mu_i^+$ hold. If $|I|^M = \kappa_d$ and $P = \mathrm{Fn}(I, \{0,1\}, \aleph_0)$, then $Z_d''(\kappa_1)$ holds.

Proof sketch

- Because the principle Z_d is defined topologically, and all trees are homeomorphic to the Cantor set, it is sufficient to prove the assertion in the case of full binary trees.
- We will consider the poset of all finite functions from κ_d into $2^{<\omega}$, which gives the same forcing. Also, let $\theta_1 = \kappa_1$ and $\theta_{i+1} = \kappa_{i+1} \setminus \kappa_i$ for $i < d$. Thus, the set κ_d is a disjoint union of the sets $\theta_1, \ldots, \theta_d$, where $|\theta_i|^M = \kappa_i$ holds for all i.
- A generic object will represent a family of generic branches of full binary trees, which are indexed by ordinals from the set κ_d.

Proof sketch

For any $\sigma \in \theta_i$ define we $\dot{b}$ as follows:

$$\dot{b}(\sigma) = \{(q, (n, k)\check{}) \mid \sigma \in \mathrm{dom}(q) \wedge n \in \mathrm{dom}(q(\sigma)) \wedge q(\sigma)(n) = k\}.$$

In other words, $\dot{b}(\sigma)$ is the name for the generic branch with index σ. Let $\varphi(f)$ be the following formula:

$$f : (2^\omega)^d \longrightarrow \kappa_1 \text{ is bounded and there is no}$$

somwhere DDF-subset of the set $(2^\omega)^d$ on which f is a constant.

Let's assume that $M[G] \models (\exists f)\varphi(f)$ holds. Let us choose $p_0 \in G$ and $\dot{f} \in M^P$ such that

$$M \models p_0 \Vdash \varphi(\dot{f})$$

holds.

Proof sketch

To each choice $\sigma_1 \in \theta_1, \ldots, \sigma_d \in \theta_d$ we can associate the condition $p(\bar{\sigma}) \leqslant p_0$ and $k(\bar{\sigma}) \in \mu$ such to be valid

$$M \models p(\bar{\sigma}) \Vdash f(\dot{b}(\sigma_1), \ldots, \dot{b}(\sigma_d)) = (k(\sigma))\check{}\,,$$

$$\{\sigma_1, \ldots, \sigma_d\} \subseteq \mathrm{dom}(p(\bar{\sigma})).$$

Let $c_i(\bar{\sigma})$ be the information that the condition $p(\bar{\sigma})$ carries about the infinite sequence of zeros and ones at the position σ_i.
The condition $p(\bar{\sigma})$ could be chosen so that all of $c_1(\bar{\sigma}), \ldots, c_d(\bar{\sigma})$ have the same domain $l(\bar{\sigma}) \in \omega$.

Proof sketch

According to the previous lemmas, $H_1 \subseteq \theta_1, \ldots, H_d \subseteq \theta_d$ can be chosen so that the following holds:

- For every i it holds $|H_i| = \kappa_i$.
- The functions k, c_i and l on the set $H_1 \times \cdots \times H_d$ are constantly equal to some values that we will mark with the same labels as those functions.
- All conditions from the set $p[H_1 \times \cdots \times H_d]$ are compatible.

Proof sketch

By the multidimensional Δ-system lemma, the sets $H_1, \ldots, H_d$ could be chosen so that there are $r_1, \ldots, r_d$ from M such that the following holds:

- $(p, H_1, \ldots, H_d, r_1, \ldots, r_d)$ is a Δ^d-system,
- $|r_1| < \aleph_0$ and $\bigwedge_{i=2}^{d} |r_i|^M = \kappa_{i-1}$.

With the symbolic from the definition of the Δ^d-system, we will denote by $F_1(q_0), \ldots, F_d(q_0)$ the sets $F_1, \ldots, F_d$ corresponding to the set $q = \mathrm{dom}(q_0)$ for a given condition q_0.
We will denote the set $H_1 \times \cdots \times H_d$ by H.

Proof sketch

Let's define

$$\dot{T} = \{(p(\bar{\sigma}), (\sigma_1, \ldots, \sigma_d)^{\check{}}) \mid \vec{\sigma} \in H\}.$$

Obviously the following holds:

$$(M \models q \Vdash (\sigma_1, \ldots, \sigma_d)^{\check{}} \in \dot{T}) \Leftrightarrow (\vec{\sigma} \in H, \wedge\, q \leqslant p(\bar{\sigma}))$$

Proof sketch

Let us prove the following:

$$M \models r_1 \Vdash_P (\forall \vec{\sigma} \in \dot{T}) \dot{f}(\dot{b}(\sigma_1), \ldots, \dot{b}(\sigma_d)) = \check{k}.$$

Otherwise, there exist $\vec{\sigma} \in H$ and $q_0 \leqslant r_1$ such that

$$M \models q_0 \Vdash_P f(\dot{b}(\sigma_1), \ldots, \dot{b}(\sigma_d)) \neq \check{k},$$

$$M \models q_0 \Vdash_P (\check{\sigma}_1, \ldots, \check{\sigma}_d) \in \dot{T}$$

holds. The last formula means that $q_0 \leqslant p(\bar{\sigma})$ and therefore

$$M \models q_0 \Vdash_P f(\dot{b}(\sigma_1), \ldots, \dot{b}(\sigma_d)) = \check{k},$$

which is a contradiction.

Proof sketch

Let us define

$$[s] = \{x \in 2^\omega \mid s \subseteq x\}, \quad s \in 2^{<\omega}.$$

Let us prove the following:

$$M \models r_1 \Vdash_P \{\dot{b}(\sigma_1) \mid (\exists \sigma_2, \ldots, \sigma_d)\vec{\sigma} \in \dot{T}\} \text{ is dense above } \check{c}_1.$$

Otherwise, there exist $s \in 2^{<\omega}$ and $q_0 \leqslant r_1$ such that $c_1 \subseteq s$ and

$$M \models q_0 \Vdash_P \{\dot{b}(\sigma_1) \mid (\exists \sigma_2, \ldots, \sigma_d)\vec{\sigma} \in \dot{T}\} \cap [\check{s}] = \check{\emptyset}$$

holds. Let us choose any $\vec{\tau}$ from the set H such that $\tau_1 \notin \mathrm{dom}(q_0)$ and $\bigwedge_{i=1}^{d} \tau_i \notin F_i(q_0)$ hold. From $\mathrm{dom}(p(\bar{\tau})) \cap \mathrm{dom}(q_0) \subseteq \mathrm{dom}(r_1)$ and $p(\bar{\tau}), q_0 \leqslant r_1$ we can conclude that $p(\bar{\tau}) || q_0$ holds.

Proof sketch

The information that the condition $p(\bar{\tau})$ contains about the Cohen's real in place τ_1 is c_1. By the choice of the element τ_1, the condition q_0 contains no information about this Cohen's real. Let us denote by q_1 the greatest condition bellow the condition $p(\bar{\tau})$ so that q_1 contains the information s about Cohen's real at place τ_1. Then $q_1 \leqslant p(\bar{\tau})$ and $q_1 || q_0$ is valid. Let us choose the condition q_2 so that $q_2 \leqslant q_0, q_1$ holds. Due to $q_2 \leqslant p(\bar{\tau})$,

$$M \models q_2 \Vdash_P (\check{\tau}_1, \ldots, \check{\tau}_d) \in \dot{T}$$

holds.

Because of $q_2 \leqslant q_1$ it holds

$$M \models q_2 \Vdash_P \dot{b}(\tau_1) \in [\check{s}],$$

which contradicts $q_2 \leqslant q_0$ and the choice of q_0.

Proof sketch

Let $k < d$ be arbitrary. Let us define the name $\dot{C}$ as

$$\dot{C} = \{\dot{b}(\sigma_{k+1}) \,|\, (\exists \sigma_{k+2}, \ldots, \sigma_d)(\forall (\sigma_1, \ldots, \sigma_k) \in F)\vec{\sigma} \in \dot{T}\}$$

if $k < d - 1$ holds and

$$\dot{C} = \{\dot{b}(\sigma_d) \,|\, (\forall (\sigma_1, \ldots, \sigma_{d-1}) \in F)\vec{\sigma} \in \dot{T}\}$$

if $k = d - 1$ holds,
Let us prove that for every finite subset F of the set

$$\{(\dot{b}(\sigma_1), \ldots, \dot{b}(\sigma_k)) \,|\, (\exists \sigma_{k+1}, \ldots, \sigma_d)\vec{\sigma} \in \dot{T}\}$$

and that in model M the condition r_1 forces that the set $\dot{C}$ is dense above c_{k+1}.

Proof sketch

Otherwise, there are $m \in \omega$, $q_0 \leqslant r$, $s \in 2^{<\omega}$ and τ_i^j for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant m$ such that $s \supseteq c_{k+1}$, in model M the condition q_0 forces that the set

$$\{\dot{b}(\sigma_{k+1}) \mid \bigwedge_{j=1}^{m} (\exists \sigma_{k+2}, \ldots \sigma_d)(\check{\tau}_1^j, \ldots, \check{\tau}_k^j, \sigma_{k+1}, \ldots, \sigma_d) \in \dot{T}\},$$

intersects $[\check{s}]$ and that the same condition forces that

$$(\check{\tau}_1^1, \ldots, \check{\tau}_d^1), \ldots, (\check{\tau}_1^m, \ldots, \check{\tau}_d^m) \in \dot{T}$$

holds. The last formula means that

$$q_0 \leqslant p(\tau_1^1, \ldots, \tau_d^1), \ldots, p(\tau_1^m, \ldots, \tau_d^m)$$

holds.

Proof sketch

Let's choose $\tau_{k+1} \in H_{k+1} \setminus F_{k+1}(q_0), \ldots, \tau_d \in H_d \setminus F_d(q_0)$ so $\tau_{k+1} \notin \mathrm{dom}(q_0)$ holds. For each $j \in \{1, \ldots, m\}$,

$$\mathrm{dom}(p(\tau_1^j, \ldots, \tau_k^j, \tau_{k+1}, \ldots, \tau_d)) \cap \mathrm{dom}(q_0) \subseteq \mathrm{dom}(r_{k+1}),$$

holds, and therefore

$$\begin{aligned} &\mathrm{dom}(p(\tau_1^j, \ldots, \tau_k^j, \tau_{k+1}, \ldots, \tau_d)) \cap \mathrm{dom}(q_0) \\ &= \mathrm{dom}(p(\tau_1^j, \ldots, \tau_k^j, \tau_{k+1}, \ldots, \tau_d)) \cap \mathrm{dom}(r_{k+1}) \cap \mathrm{dom}(q_0) \\ &= \mathrm{dom}(p(\tau_1^j, \ldots, \tau_d^j)) \cap \mathrm{dom}(r_{k+1}) \cap \mathrm{dom}(q_0), \end{aligned}$$

which together with $q_0 \leqslant p(\tau_1^j, \ldots, \tau_d^j)$ and

$$p(\tau_1^j, \ldots, \tau_k^j, \tau_{k+1}, \ldots, \tau_d) \restriction \mathrm{dom}(r_{k+1}) = p(\tau_1^j, \ldots, \tau_d^j) \restriction \mathrm{dom}(r_{k+1})$$

implies that $p(\tau_1^j, \ldots, \tau_k^j, \tau_{k+1}, \ldots, \tau_d) || q_0$ holds.

Proof sketch

For the condition q_1 defined as

$$q_1 = \inf \{ p(\tau_1^j, \dots, \tau_k^j, \tau_{k+1} \dots, \tau_d) \mid j \in \{1, \dots, m\} \},$$

$q_1 || q_0$ holds. The information that q_1 contains about Cohen's real at position τ_{k+1} is c_{k+1}, while from $\tau_{k+1} \notin \mathrm{dom}(q_0)$ we conclude that q_0 does not contains any information about it. Let q_2 be the greatest condition less than the condition q_1 so that q_2 contains the information s about the Cohen real number at position τ_2. Then, $q_2 \leqslant q_1$ and $q_2 || q_0$ hold, as well as in model M the condition q_2 forces that

$$\dot{b}(\tau_{k+1}) \in [\check{s}]$$

and the same condition forces that

$$(\tau_1^1, \dots, \tau_k^1, \tau_{k+1}, \dots, \tau_d), \dots, (\tau_1^m, \dots, \tau_k^m, \tau_{k+1}, \dots, \tau_d) \in \dot{T},$$

which is a contradiction.

A generalization of Baire's category theorem

Note that for each uncountable cardinal κ, the principle $Z_1''(\kappa)$ is a generalization of Baire's category theorem to uncountable partitions of the space $\mathbb{R}^n$ of cardinality less than κ.

Let M be a countable transitive model and let λ_n be a sequence in M of regular cardinals in M such that for each $i \in \omega$ there exists some $\mu_i \geqslant 2^{\lambda_i}$ such that $\mu_i^{\lambda_i} = \mu_i$ and $\lambda_{i+1} = \mu_i^+$. Let λ be a regular cardinal in M, which is greater than λ_i for every i. If $|I|^M = \lambda$ and $P = \mathrm{Fn}(I, \{0,1\}, \aleph_0)$, then $M[G] \models Z''(\lambda_1)$.

Proof

Otherwise there exists $d \geqslant 3$ such that $M[G] \models \neg Z''_d(\lambda)$. Let $\kappa_i = \lambda_i$ for $i < d$ and $\kappa_d = \lambda$. Applying the previous theorem, we can conclude concluded that $M[G] \models Z''_d(\lambda)$ holds, which is a contradiction.

Definition of a simple cardinal

A simple cardinal is an infinite cardinal, which can be written as a finite expression in which only constants from the set $\omega \cup \{\omega\}$ and the following operations appear:

$\alpha \mapsto \omega_\alpha$,

adding, multiplication and exponentiation of ordinals.

Absoluteness theorem

Let κ be any simple cardinal. The following

$$(ZFC + Z''(\kappa) \vdash F) \Rightarrow (ZFC \vdash F)$$

holds for every regular statement F.

Thank you for your attention.