Applications of the open dihypergraph dichotomy for generalized Baire spaces

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Based on a joint project with Philipp Schlicht.

- Philipp Schlicht, Dorottya Sziráki:
  The open dihypergraph dichotomy for generalized Baire spaces,
  97 pages, in preparation.

Motivation: Ramsey theory

Theorem (Ramsey): Every infinite graph has an infinite clique or anticlique.

Can "infinite" be replaced by "uncountable"?

- Sierpinski's counterexample (1937): a graph on $\mathbb{R}$ with no uncountable cliques or anticliques.
- Galvin (1968): Ramsey's theorem holds for all clopen graphs on $\mathbb{R}$; in fact, there is a perfect clique or anticlique for all such graphs.
- Blass (1981): A generalization to Borel n-hypergraphs.

The open graph dichotomy

An open graph is an open subset of $X \times X$ which is symmetric and irreflexive.

Definition (Feng, Todorčević)

$\mathrm{OGD}_\omega(X)$ states that for any open graph G on X, either

- G has a perfect clique, or
- G has an ω-coloring.

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$\mathrm{OGD}_\omega(X)$ *holds for analytic subsets of* ${}^\omega\omega$. *In Solovay's model, it holds for all subsets of* ${}^\omega\omega$.

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$\mathrm{OGD}_\omega(X)$ *holds for analytic subsets of* ${}^\omega\omega$*. In Solovay's model, it holds for all subsets of* ${}^\omega\omega$*.*

$\mathrm{OGD}_\omega(X)$ implies the perfect set property for X: either X is countable or X has a perfect subset.

A high dimensional version

Recently, dichotomies for graphs and dihypergraphs have emerged which imply several old and new theorems in descriptive set theory.

Carroy, Miller and Soukup (2020) found such an infinite dimensional version of the open graph dichotomy.

A κ-dihypergraph on X is a set of nonconstant sequences in ${}^{\kappa}X$.

A high dimensional version

We fix the box topology on ${}^{\omega}X$ with basic open sets $\prod_{i<\omega} U_i$, where each U_i is open in X.

Definition (Carroy, Miller, Soukup 2020)

$\mathrm{ODD}^{\omega}_{\omega}(X)$: for any box-open ω-dihypergraph H on X, either

- H has an ω-coloring,
- or H contains a “large” dihyperpgraph.

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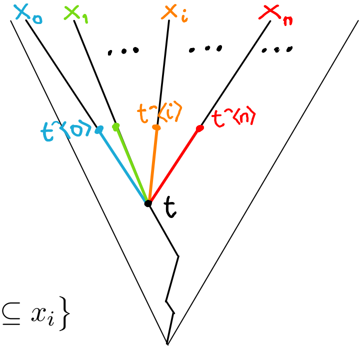
- H has an ω-coloring,
- or there is a continuous homomorphism from $\mathbb{H}_{\omega_\omega}$ to H.

$$\mathbb{H}_{\omega_\omega} = \left\{ \overline{x} \in {}^{\omega}({}^{\omega}\omega) : \exists t \in {}^{<\omega}\omega \ \forall i \in \omega \ \ t^\frown \langle i \rangle \subseteq x_i \right\}$$



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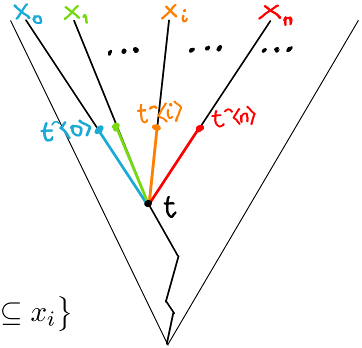


Fact: $\mathrm{ODD}^{\omega}_{\omega}(X)$ implies the open graph dichotomy $\mathrm{OGD}_{\omega}(X)$.

A high dimensional version

Theorem (Carroy, Miller, Soukup 2020)

- $\mathrm{ODD}^{\omega}_{\omega}(X)$ *holds for analytic subsets* X *of* ${}^{\omega}\omega$.
- *Assuming* AD, *it holds for all subsets of* ${}^{\omega}\omega$.

Applications

Theorem (Carroy, Miller, Soukup 2020)

For any family Γ *of subsets of* ${}^{\omega}\omega$, $\mathrm{ODD}^{\omega}_{\omega}(\Gamma)$ *implies:*

- *the Hurewicz dichotomy: any* $X \in \Gamma$ *either contains a closed homeomorphic copy of* ${}^{\omega}\omega$ *or is contained in a* K_{σ} *set;*
- *the Kechris-Louveau-Woodin dichotomy on separating subsets in* Γ *from arbitrary subsets by an* F_{σ} *set;*
- *the Jayne-Rogers theorem on piecewise continuous functions* f *with closed pieces such that* $\mathrm{dom}(f) \in \Gamma$*.*

Generalized Baire spaces

κ always denotes an uncountable cardinal with $\kappa^{<\kappa} = \kappa$.

Definitions are analogous:

- The κ-Baire space ${}^\kappa\kappa$ is the set of functions $f : \kappa \to \kappa$, with the bounded topology: basic open sets are

$$N_s = \{x \in {}^\kappa\kappa : s \subset x\} \qquad \text{for all } s \in {}^{<\kappa}\kappa.$$

- The κ-Cantor space ${}^\kappa 2$ has the subspace topology.
- κ-Borel sets are generated from open sets by closing under intersections and unions of size κ and complementation.

Regularity properties

It is consistent relative to an inaccessible cardinal that analogues of

- the perfect set property and the Hurewicz dichotomy hold for definable subsets of ${}^{\kappa}\kappa$ (Schlicht 2017; Lücke, Motto–Ros, Schlicht);

  By definable, we always mean definable from a parameter in ${}^{\kappa}\mathrm{Ord}$.

- the open graph dichotomy holds for κ-analytic subsets of ${}^{\kappa}\kappa$ (Sz. 2018).

  A κ-analytic set is a continuous image of a closed subset.

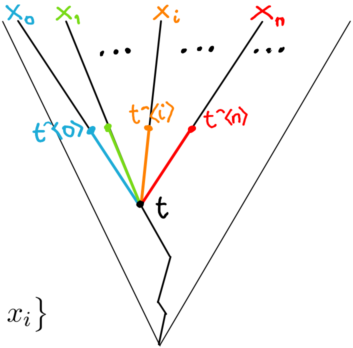
ODD for the κ-Baire space

Definition

$\mathrm{ODD}_\kappa^\kappa(X)$: for any box-open κ-dihypergraph H on X, either

- H has a κ-coloring,
- or there is a continuous homomorphism $f : {}^\kappa\kappa \to X$ from $\mathbb{H}_{{}^\kappa\kappa}$ to H.

$$\mathbb{H}_{{}^\kappa\kappa} = \left\{ \overline{x} \in {}^\kappa({}^\kappa\kappa) : \exists t \in {}^{<\kappa}\kappa \; \forall i \in \kappa \;\; t^\frown\langle i\rangle \subseteq x_i \right\}$$



$\mathrm{ODD}_\kappa^\kappa(X, H)$ states that this holds for H.

ODD for definable subsets

By definable, we always mean definable from a parameter in ${}^{\kappa}\mathrm{Ord}$.

Theorem (Schlicht, Sz. 2021)

After a Lévy-collapse of λ to κ^+, for all definable subsets X of ${}^{\kappa}\kappa$:

- $\mathrm{ODD}^{\kappa}_{\kappa}(X)$ *holds if λ is Mahlo in the ground model,*

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- $\mathrm{ODD}^{\kappa}_{\kappa}(X, H)$ *holds for all definable box-open* κ*-dihypergraphs* H *on* X *if* λ *is inaccessible in the ground model.*

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- $\mathrm{ODD}^{\kappa}_{\kappa}(X, H)$ *holds for all definable box-open κ-dihypergraphs H on X if λ is inaccessible in the ground model.*

A simplified version of the proof works for ω. So in Solovay's model, $\mathrm{ODD}^{\omega}_{\omega}(X)$ holds for all $X \subseteq {}^{\omega}\omega$.

All the applications of Carroy-Miller-Soukup for definable subsets of ${}^{\omega}\omega$ are consistent relative to an inaccessible or Mahlo cardinal.

Applications

All the applications of Carroy-Miller-Soukup for definable subsets of ${}^{\omega}\omega$ are consistent relative to an inaccessible or Mahlo cardinal.

New proof of the consistency of the perfect set property and the open graph dichotomy for definable subsets of ${}^{\kappa}\kappa$.

Applications

Theorem (Schlicht, Sz. 2022)

For any family Γ *of subsets of* ${}^{\kappa}\kappa$, $\mathrm{ODD}^{\kappa}_{\kappa}(\Gamma)$ *implies analogues of:*

- *the Hurewicz dichotomy: any* $X \in \Gamma$ *either contains a closed homeomorphic copy of* ${}^{\kappa}\kappa$ *or is contained in a union of* κ *many* κ*-compact sets;*
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The analogues of Hurewicz dichotomy and the Jayne-Rogers theorem already follow from a definable version of $\mathrm{ODD}_{\kappa}^{\kappa}(\Gamma)$:

$\mathrm{ODD}_{\kappa}^{\kappa}(X, H{\restriction}X)$ holds for all definable box-open κ-dihypergraphs H on ${}^{\kappa}\kappa$.

Hence these are these are consistent for definable subsets of ${}^{\kappa}\kappa$ relative to an inaccessible cardinal $\lambda > \kappa$.

The Kechris-Louveau-Woodin dichotomy for definable subsets of ${}^{\kappa}\kappa$ is consistent relative to a Mahlo cardinal $\lambda > \kappa$.

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Applications: Assymetric Baire property

Definition: $X \subseteq {}^\kappa\kappa$ has the κ-Baire property if there is an open set $U \subseteq {}^\kappa\kappa$ such that $X \triangle U$ is κ-meager.

The κ-Baire property fails for some κ-analytic subset of ${}^\kappa\kappa$ (Halko, Shelah).

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Definition: X has the asymmetric Baire property if the the Banach-Mazur game of length κ for X is determined.

In this game, players $\mathbf{I}$ and $\mathbf{II}$ play a strictly increasing sequence $\langle s_\alpha : \alpha < \kappa \rangle$ in ${}^{<\kappa}\kappa$. $\mathbf{I}$ plays in all even rounds (including limit rounds). $\mathbf{I}$ wins if $\bigcup_{\alpha<\kappa} s_\alpha \in X$.

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- κ-Baire-property $\Rightarrow$ asymmetric Baire property for subsets of ${}^{\kappa}\kappa$.
- The two properties are equivalent for ω.

Applications: Assymetric Baire property

Theorem (Schlicht, Sz. 2022)

- *The definable version of* $\mathrm{ODD}_\kappa^\kappa(X)$ *implies the asymmetric Baire property for* $X \subseteq {}^\kappa\kappa$.
- *For* ω *instead of* κ, *it implies the Baire property.*

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New proof of the consistency of

- the asymmetric Baire property for definable subsets of ${}^{\kappa}\kappa$ (Schlicht 2017);
- the Baire property for definable subsets of ${}^{\omega}\omega$ (Solovay 1970).

Applications: Väänänen's perfect set game

By Cantor-Bendixson analysis, any topological space can be decomposed into a scattered subset and a crowded subset by iteratively removing isolated points. If the space has a countable base, then the scattered part is countable.

Väänänen adapted this to subsets $X \subseteq {}^{\kappa}\kappa$ via a game $\mathcal{G}_{\kappa}(X, x)$ of length κ.

Väänänen's perfect set game

Definition (Väänänen 1991)

The game $\mathcal{G}_\kappa(X, x)$ of length κ for subsets X of ${}^\kappa\kappa$ is played as follows:

I plays a strictly decreasing sequence $\langle U_\alpha : 1 \leq \alpha < \kappa \rangle$ of open subsets of X so that $x_\beta \in U_{\beta+1}$ if $\beta + 1 < \kappa$ and $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$ in limit rounds α.

II plays an injective sequence $\langle x_\alpha : \alpha < \kappa \rangle$ with $x_0 = x$ and $x_\alpha \in U_\alpha$.

I		U_1		$\dots$		U_α		$\dots$
II	$x_0 = x$		x_1		$\dots$		x_α		$\dots$

Each player loses immediately if they do not follow these rules.
If both follow the rules in all rounds, then **II** wins.

- X is κ-scattered $:\Leftrightarrow$ **I** wins $\mathcal{G}_\kappa(X, x)$ for all $x \in X$.
- X is κ-crowded $:\Leftrightarrow$ **II** wins $\mathcal{G}_\kappa(X, x)$ for all $x \in X$

  $\Leftrightarrow$ the closure of X is κ-perfect.

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Theorem (Schlicht, Sz. 2022)

$\mathrm{ODD}^\kappa_\kappa({}^\kappa\kappa)$ *implies that all subsets of* ${}^\kappa\kappa$ *can be decomposed into a* κ*-scattered subset of size* $\leq \kappa$ *and a* κ*-crowded subset.*

In particular, $\mathcal{G}_\kappa(X, x)$ *is determined for all* $X \subseteq {}^\kappa\kappa$ *and* $x \in X$.

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Hence this property for all subsets of ${}^\kappa\kappa$ is consistent relative to a Mahlo cardinal.

This strenghtens Väänänen's result (1991) on the consistency of this property for closed subsets relative to a measurable cardinal.

Thank you!