Colorings and subdivisions: a partition principle for higher limits

Jeffrey Bergfalk
University of Barcelona

Young Set Theory Workshop

Novi Sad
August 2022

the goal

the goal

The focus of this talk will be the recent work

A descriptive approach to higher derived limits,

joint with Nathaniel Bannister, Justin Tatch Moore, and Stevo Todorcevic (`arXiv` 2022).

the goal

The focus of this talk will be the recent work

A descriptive approach to higher derived limits,

joint with Nathaniel Bannister, Justin Tatch Moore, and Stevo Todorcevic (`arXiv` 2022).

Although this isn't a particularly simple work, its idea and motivation, I think, are, and it's these that I'll aim above all to communicate in the next hour.

the plan

the plan

Here's my plan:

the plan

Here's my plan:

1. Some of the prehistory (i.e., *motivation*) of this paper.

the plan

Here's my plan:

1. Some of the prehistory (i.e., *motivation*) of this paper.
2. The *simple idea*: the partition principles PH_n ($n \in \omega$), and their place in this history.

the plan

Here's my plan:

1. Some of the prehistory (i.e., *motivation*) of this paper.
2. The *simple idea*: the partition principles PH_n $(n \in \omega)$, and their place in this history.
3. Some warm-up and practice with these principles in the setting of the ordinals.

the plan

Here's my plan:

1. Some of the prehistory (i.e., *motivation*) of this paper.
2. The *simple idea*: the partition principles PH_n ($n \in \omega$), and their place in this history.
3. Some warm-up and practice with these principles in the setting of the ordinals.
4. A simplicial perspective from which these principles *really are simple*.

the plan

Here's my plan:

1. Some of the prehistory (i.e., *motivation*) of this paper.
2. The *simple idea*: the partition principles PH_n ($n \in \omega$), and their place in this history.
3. Some warm-up and practice with these principles in the setting of the ordinals.
4. A simplicial perspective from which these principles *really are simple*.
5. A return to ${}^\omega\omega$.

in the beginning

in the beginning

In 1988, Mardešić and Prasolov posed the question of *whether strong homology* $\bar{\mathrm{H}}_*$ *is additive.*

in the beginning

In 1988, Mardešić and Prasolov posed the question of *whether strong homology* $\bar{\mathrm{H}}_*$ *is additive.*

Let's unpack these terms. First, *additivity*:

in the beginning

In 1988, Mardešić and Prasolov posed the question of *whether strong homology* $\bar{\mathrm{H}}_*$ *is additive.*

Let's unpack these terms. First, *additivity*:

Definition (Milnor 1962)

A homology theory H_* is *additive on the class* $\mathcal{C}$ *of topological spaces* if for all p and $\{X_\alpha \mid \alpha \in A\} \subseteq \mathcal{C}$ with $\coprod_{\alpha \in A} X_\alpha$ in $\mathcal{C}$, the map

$$\bigoplus_{\alpha \in A} \mathrm{H}_p(X_\alpha) \to \mathrm{H}_p(\coprod_{\alpha \in A} X_\alpha)$$

induced by the inclusions $X_\alpha \hookrightarrow \coprod_{\alpha \in A} X_\alpha$ is an isomorphism.

in the beginning

In 1988, Mardešić and Prasolov posed the question of *whether strong homology* $\bar{\mathrm{H}}_*$ *is additive.*

Let's unpack these terms. First, *additivity*:

Definition (Milnor 1962)

A homology theory H_* is *additive on the class* $\mathcal{C}$ *of topological spaces* if for all p and $\{X_\alpha \mid \alpha \in A\} \subseteq \mathcal{C}$ with $\coprod_{\alpha \in A} X_\alpha$ in $\mathcal{C}$, the map

$$\bigoplus_{\alpha \in A} \mathrm{H}_p(X_\alpha) \to \mathrm{H}_p(\coprod_{\alpha \in A} X_\alpha)$$

induced by the inclusions $X_\alpha \hookrightarrow \coprod_{\alpha \in A} X_\alpha$ is an isomorphism.

Second, *strong homology*:

a sketch of the idea of strong homology

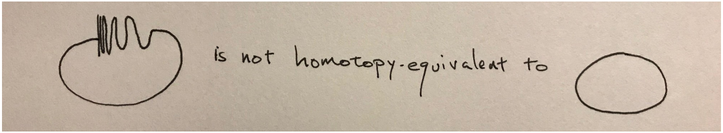
A homology theory, axiomatically understood, can behave in only one way on the category of finite CW-complexes; additivity determines its extension to the category HCW of spaces having the homotopy type of (possibly infinite) CW complexes.

a sketch of the idea of strong homology

A homology theory, axiomatically understood, can behave in only one way on the category of finite CW-complexes; additivity determines its extension to the category HCW of spaces having the homotopy type of (possibly infinite) CW complexes. Other considerations may arise when we endeavor to extend it further.

a sketch of the idea of strong homology

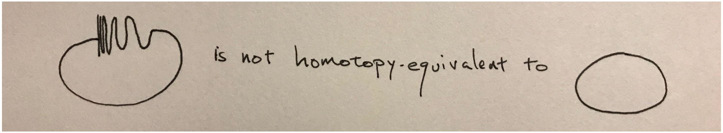
A homology theory, axiomatically understood, can behave in only one way on the category of finite CW-complexes; additivity determines its extension to the category HCW of spaces having the homotopy type of (possibly infinite) CW complexes. Other considerations may arise when we endeavor to extend it further. For example,



but it has its *shape*, in the sense that both these figures divide the plane, and that their systems of neighborhoods are structurally equivalent.

a sketch of the idea of strong homology

A homology theory, axiomatically understood, can behave in only one way on the category of finite CW-complexes; additivity determines its extension to the category HCW of spaces having the homotopy type of (possibly infinite) CW complexes. Other considerations may arise when we endeavor to extend it further. For example,



but it has its *shape*, in the sense that both these figures divide the plane, and that their systems of neighborhoods are structurally equivalent. Among the several virtues of *strong homology* $\bar{\mathrm{H}}_*$ are its strong shape-invariance: it is the pre-eminent homology theory with this feature which coincides with the classical theories on HCW.

meanings of the question

meanings of the question

The additivity question, then, is one of whether strong homology has additional classical and desirable properties; it is even a question about the very compatibility of these properties with shape-invariance on general classes of topological spaces.

meanings of the question

The additivity question, then, is one of whether strong homology has additional classical and desirable properties; it is even a question about the very compatibility of these properties with shape-invariance on general classes of topological spaces. Some care in the posing of these questions is necessary, due to a nonmetrizable ZFC counterexample to additivity due to Prasolov, dating to the late 90s — but the gap between that example and the sorts of spaces we'll be discussing is substantial.

meanings of the question

The additivity question, then, is one of whether strong homology has additional classical and desirable properties; it is even a question about the very compatibility of these properties with shape-invariance on general classes of topological spaces. Some care in the posing of these questions is necessary, due to a nonmetrizable ZFC counterexample to additivity due to Prasolov, dating to the late 90s — but the gap between that example and the sorts of spaces we'll be discussing is substantial. In short, in more precise form, the question that had remained open until quite recently was the following:

Can strong homology be additive on any robust class of topological spaces properly extending HCW*?*

meanings of the question

The additivity question, then, is one of whether strong homology has additional classical and desirable properties; it is even a question about the very compatibility of these properties with shape-invariance on general classes of topological spaces. Some care in the posing of these questions is necessary, due to a nonmetrizable ZFC counterexample to additivity due to Prasolov, dating to the late 90s — but the gap between that example and the sorts of spaces we'll be discussing is substantial. In short, in more precise form, the question that had remained open until quite recently was the following:

Can strong homology be additive on any robust class of topological spaces properly extending HCW*?*

What Mardešić and Prasolov had noticed around 1988 was the following fundamental impediment to additivity.

Is strong homology additive?

Mardešić and Prasolov
observed
that if strong homology $\bar{H}_*$ is <u>additive</u> then

$$\bar{H}_p(\circledcirc^{(k)}) \oplus \bar{H}_p(\circledcirc^{(k)}) \oplus \bar{H}_p(\circledcirc^{(k)}) \oplus \dots$$
$$= \bar{H}_p(\circledcirc^{(k)} \sqcup \circledcirc^{(k)} \sqcup \circledcirc^{(k)} \sqcup \dots)$$

for any $k, p \in \omega$. They then computed that for $0 < p < k$ this equates to

$$0 \oplus 0 \oplus 0 \oplus \dots$$
$$= \lim{}^{k-p} \mathbf{A}.$$

Hence the question: $\mathrm{Con}(\lim^{k-p} \mathbf{A} = 0)$?

the inverse system $\mathbf{A}$

the inverse system $\mathbf{A}$

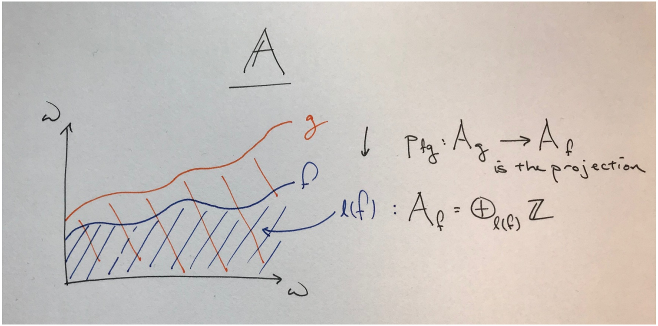
— i.e., necessary for the additivity of $\bar{\mathrm{H}}^*$ is the vanishing of the $\lim^n$ $(n > 0)$ groups of the system $\mathbf{A} = (A_f, p_{fg}, \mathcal{N})$, where

- $\mathcal{N}$ is the partial order $({}^\omega\omega, \leq)$,
- $A_f = \bigoplus_{\ell(f)} \mathbb{Z}$, where $\ell(f) = \{(i,j) \in \omega^2 \mid j \leq f(i)\}$, and
- $p_{fg} : A_g \to A_f$ is the projection map for each $f \leq g$.

the inverse system $\mathbf{A}$

— i.e., necessary for the additivity of $\bar{\mathrm{H}}^*$ is the vanishing of the $\lim^n$ $(n > 0)$ groups of the system $\mathbf{A} = (A_f, p_{fg}, \mathcal{N})$, where

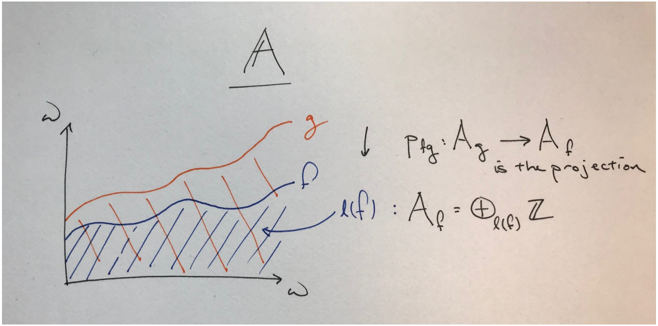
- $\mathcal{N}$ is the partial order $({}^\omega\omega, \leq)$,
- $A_f = \bigoplus_{\ell(f)} \mathbb{Z}$, where $\ell(f) = \{(i,j) \in \omega^2 \mid j \leq f(i)\}$, and
- $p_{fg} : A_g \to A_f$ is the projection map for each $f \leq g$.



the inverse system $\mathbf{A}$

— i.e., necessary for the additivity of $\bar{\mathrm{H}}^*$ is the vanishing of the $\lim^n$ $(n > 0)$ groups of the system $\mathbf{A} = (A_f, p_{fg}, \mathcal{N})$, where

- $\mathcal{N}$ is the partial order $({}^\omega\omega, \leq)$,
- $A_f = \bigoplus_{\ell(f)} \mathbb{Z}$, where $\ell(f) = \{(i,j) \in \omega^2 \mid j \leq f(i)\}$, and
- $p_{fg} : A_g \to A_f$ is the projection map for each $f \leq g$.



Observe that $\lim \mathbf{A}$ may be identified with

$$\{\varphi : \omega \times \omega \to \mathbb{Z} \mid \mathrm{supp}(\varphi) \in (\mathit{fin} \times \varnothing)\}.$$

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$ admits the following characterization, also due to Mardešić and Prasolov.

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$ admits the following characterization, also due to Mardešić and Prasolov. Here $F =^* G$ if F is equal mod finite to G on the intersection of their domains.

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$ admits the following characterization, also due to Mardešić and Prasolov. Here $F =^* G$ if F is equal mod finite to G on the intersection of their domains.

Definition

A family of functions $\Phi = \{\varphi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^\omega\omega\}$ is coherent if

$$\varphi_g - \varphi_f =^* 0$$

for all f and g in ${}^\omega\omega$.

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$ admits the following characterization, also due to Mardešić and Prasolov. Here $F =^* G$ if F is equal mod finite to G on the intersection of their domains.

Definition

A family of functions $\Phi = \{\varphi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^\omega\omega\}$ is coherent if

$$\varphi_g - \varphi_f =^* 0$$

for all f and g in ${}^\omega\omega$. Φ is trivial if there exists a $\varphi : \omega \times \omega \to \mathbb{Z}$ such that

$$\varphi - \varphi_f =^* 0$$

for all f in ${}^\omega\omega$.

$\lim^1 \mathbf{A}$

$\lim^1 \mathbf{A}$ admits the following characterization, also due to Mardešić and Prasolov. Here $F =^* G$ if F is equal mod finite to G on the intersection of their domains.

Definition

A family of functions $\Phi = \{\varphi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^\omega\omega\}$ is coherent if

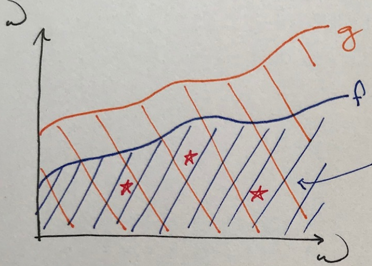
$$\varphi_g - \varphi_f =^* 0$$

for all f and g in ${}^\omega\omega$. Φ is trivial if there exists a $\varphi : \omega \times \omega \to \mathbb{Z}$ such that

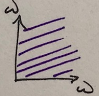
$$\varphi - \varphi_f =^* 0$$

for all f in ${}^\omega\omega$. Observe that pointwise addition endows both the collection `Coh` of coherent families of functions and the collection `Triv` of trivial families of functions with the structure of a group. $\lim^1 \mathbf{A}$ is isomorphic to the quotient `Coh`/`Triv`.

$\lim^1 \mathbb{A}$



★s mark the finitely disagreements, in $I(f)$, between φ_f and φ_g. This is <u>coherence</u>.

Does there exist a φ :  $\longrightarrow \mathbb{Z}$

agreeing mod finite with all φ_fs?

This is <u>triviality</u>.

$\lim^1 \mathbf{A}$, assuming CH

$\lim^1 \mathbf{A}$, assuming CH

By the above definition, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial.

$\lim^1 \mathbf{A}$, assuming CH

By the above definition, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial. Mardešić and Prasolov attribute the observation

$$\mathsf{CH} \Rightarrow \text{[there exists a nontrivial coherent family of functions]}$$

to Petr Simon.

$\lim^1 \mathbf{A}$, assuming CH

By the above definition, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial. Mardešić and Prasolov attribute the observation

$$\mathsf{CH} \Rightarrow [\text{there exists a nontrivial coherent family of functions}]$$

to Petr Simon. The argument is routine, modulo the following recognition:

Lemma

Let $F \subseteq {}^{\omega}\omega$ be a $<^*$-cofinal family of functions. A coherent family Φ is trivial if and only if its restriction to F is trivial.

$\lim^1 \mathbf{A}$, assuming CH

By the above definition, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial. Mardešić and Prasolov attribute the observation

$$\mathsf{CH} \Rightarrow \text{[there exists a nontrivial coherent family of functions]}$$

to Petr Simon. The argument is routine, modulo the following recognition:

Lemma

Let $F \subseteq {}^{\omega}\omega$ be a $<^*$-cofinal family of functions. A coherent family Φ is trivial if and only if its restriction to F is trivial.

Using the continuum hypothesis, then, fix an $<^*$-scale $F = \langle f_\alpha \mid \alpha < \omega_1 \rangle$ and, working upwards through F, (cofinally) define a coherent family of functions Φ which diagonalizes against all the $\aleph_1$ many possible trivializations of Φ.

$\lim^1 \mathbf{A}$, assuming CH

By the above definition, $\lim^1 \mathbf{A} = 0$ if and only if every coherent family of functions is trivial. Mardešić and Prasolov attribute the observation

$$\mathsf{CH} \Rightarrow [\text{there exists a nontrivial coherent family of functions}]$$

to Petr Simon. The argument is routine, modulo the following recognition:

Lemma

Let $F \subseteq {}^\omega\omega$ be a $<^*$-cofinal family of functions. A coherent family Φ is trivial if and only if its restriction to F is trivial.

Using the continuum hypothesis, then, fix an $<^*$-scale $F = \langle f_\alpha \mid \alpha < \omega_1 \rangle$ and, working upwards through F, (cofinally) define a coherent family of functions Φ which diagonalizes against all the $\aleph_1$ many possible trivializations of Φ. This shows $\mathsf{CH} \Rightarrow [\lim^1 \mathbf{A} \neq 0]$.

$\lim^1 \mathbf{A}$ under other assumptions

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- PFA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- PFA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- PFA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- OCA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- $\mathsf{PFA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- $\mathsf{OCA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- PFA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- OCA $\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);
- In the extension by a forcing to add at least ω_2 many Cohen reals, $\lim^1 \mathbf{A} = 0$ (Kamo 1993);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- $\mathsf{PFA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- $\mathsf{OCA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);
- In the extension by a forcing to add at least ω_2 many Cohen reals, $\lim^1 \mathbf{A} = 0$ (Kamo 1993);
- $\mathsf{MA}(\omega_1) \not\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1998);

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- $\mathsf{PFA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- $\mathsf{OCA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);
- In the extension by a forcing to add at least ω_2 many Cohen reals, $\lim^1 \mathbf{A} = 0$ (Kamo 1993);
- $\mathsf{MA}(\omega_1) \not\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1998);
- Any witness to $\lim^1 \mathbf{A} \neq 0$ is (in the natural topology) a nonanalytic set whose intersection with any compact set is F_σ (Todorcevic 1998).

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- $\mathsf{PFA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- $\mathsf{OCA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);
- In the extension by a forcing to add at least ω_2 many Cohen reals, $\lim^1 \mathbf{A} = 0$ (Kamo 1993);
- $\mathsf{MA}(\omega_1) \not\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1998);
- Any witness to $\lim^1 \mathbf{A} \neq 0$ is (in the natural topology) a nonanalytic set whose intersection with any compact set is F_σ (Todorcevic 1998).

These results remained the state of the art for roughly the next twenty years.

$\lim^1 \mathbf{A}$ under other assumptions

This work by Mardešić, Prasolov, and Simon cued a rapid succession of results and refinements:

- $\mathsf{PFA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Dow, Simon, Vaughan 1988);
- $[\mathfrak{d} = \aleph_1] \Rightarrow [\lim^1 \mathbf{A} \neq 0]$ (Dow, Simon, Vaughan 1988);
- $\mathsf{OCA} \Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1989);
- Any ordertype-ω_1 $<^*$-increasing family of functions from ω to ω indexes a nontrivial coherent family (Todorcevic 1991);
- In the extension by a forcing to add at least ω_2 many Cohen reals, $\lim^1 \mathbf{A} = 0$ (Kamo 1993);
- $\mathsf{MA}(\omega_1) \not\Rightarrow [\lim^1 \mathbf{A} = 0]$ (Todorcevic 1998);
- Any witness to $\lim^1 \mathbf{A} \neq 0$ is (in the natural topology) a nonanalytic set whose intersection with any compact set is F_σ (Todorcevic 1998).

These results remained the state of the art for roughly the next twenty years. As we've seen, however, the additivity of strong homology would require that $\lim^n \mathbf{A} = 0$ for *all* $n > 0$.

$$\lim^2 \mathbf{A}$$

$\lim^2 \mathbf{A}$

Definition and Theorem

A family $\Phi = \{\varphi_{fg} : \ell(f \wedge g) \to \mathbb{Z} \mid f, g \in {}^{\omega}\omega\}$ *is* alternating *if* $\varphi_{fg} = -\varphi_{gf}$ *for all* $f, g \in {}^{\omega}\omega$.

$\lim^2 \mathbf{A}$

Definition and Theorem

A family $\Phi = \{\varphi_{fg} : \ell(f \wedge g) \to \mathbb{Z} \mid f, g \in {}^\omega\omega\}$ *is* alternating *if* $\varphi_{fg} = -\varphi_{gf}$ *for all* $f, g \in {}^\omega\omega$.

An alternating Φ *as above is* 2-coherent *if*

$$\varphi_{fh} - \varphi_{gh} =^* \varphi_{fg}$$

for all $f, g, h \in {}^\omega\omega$.

$\lim^2 \mathbf{A}$

Definition and Theorem

A family $\Phi = \{\varphi_{fg} : \ell(f \wedge g) \to \mathbb{Z} \mid f, g \in {}^{\omega}\omega\}$ *is* alternating *if* $\varphi_{fg} = -\varphi_{gf}$ *for all* $f, g \in {}^{\omega}\omega$*.*

An alternating Φ *as above is* 2-coherent *if*

$$\varphi_{fh} - \varphi_{gh} =^* \varphi_{fg}$$

for all $f, g, h \in {}^{\omega}\omega$*.*

Φ *is* 2-trivial *if there exists a* $\Psi = \{\psi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^{\omega}\omega\}$ *such that*

$$\psi_f - \psi_g =^* \varphi_{fg}$$

for all $f, g \in {}^{\omega}\omega$*.*

$\lim^2 \mathbf{A}$

Definition and Theorem

A family $\Phi = \{\varphi_{fg} : \ell(f \wedge g) \to \mathbb{Z} \mid f, g \in {}^\omega\omega\}$ *is* alternating *if* $\varphi_{fg} = -\varphi_{gf}$ *for all* $f, g \in {}^\omega\omega$.

An alternating Φ *as above is* 2-coherent *if*

$$\varphi_{fh} - \varphi_{gh} =^* \varphi_{fg}$$

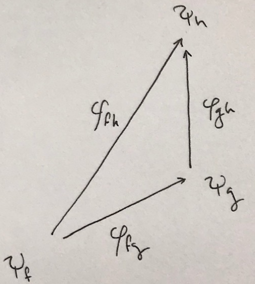
for all $f, g, h \in {}^\omega\omega$.

Φ *is* 2-trivial *if there exists a* $\Psi = \{\psi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^\omega\omega\}$ *such that*

$$\psi_f - \psi_g =^* \varphi_{fg}$$

for all $f, g \in {}^\omega\omega$.

$\lim^2 \mathbf{A} = 0$ *iff every 2-coherent family is 2-trivial.*



$$\varphi_{fg} + \varphi_{gh} =^* \varphi_{fh}$$

for all $f, g, h \in \omega^\omega$.

Is this because there exist ψ_f, ψ_g, ψ_h such that

$$\psi_g - \psi_f =^* \varphi_{fg}$$

$$\psi_h - \psi_g =^* \varphi_{gh}$$

$$\psi_h - \psi_f =^* \varphi_{fh}$$

?

Locally, it must be.

... but globally?

more recent results

more recent results

Theorem (B. 2017)

The Proper Forcing Axiom implies that $\lim^2 \mathbf{A} \neq 0$.

more recent results

Theorem (B. 2017)

The Proper Forcing Axiom implies that $\lim^2 \mathbf{A} \neq 0$.

Theorem (B., Lambie-Hanson 2019)

Let κ *be a weakly compact cardinal and let* $\mathbb{P}$ *denote a length-*κ *finite-support iteration of Hechler forcings. Then*

$$V^{\mathbb{P}} \models \text{“}\lim^n \mathbf{A} = 0 \text{ for all } n > 0.\text{”}$$

more recent results

Theorem (B. 2017)

The Proper Forcing Axiom implies that $\lim^2 \mathbf{A} \neq 0$.

Theorem (B., Lambie-Hanson 2019)

Let κ be a weakly compact cardinal and let $\mathbb{P}$ denote a length-κ finite-support iteration of Hechler forcings. Then

$$V^{\mathbb{P}} \models \text{“}\lim^n \mathbf{A} = 0 \text{ for all } n > 0.\text{”}$$

Theorem (Bannister, B., Moore 2020)

In the model $V^{\mathbb{P}}$ appearing above, strong homology is additive on the class of locally compact separable metric spaces.

more recent results

Theorem (B. 2017)

The Proper Forcing Axiom implies that $\lim^2 \mathbf{A} \neq 0$.

Theorem (B., Lambie-Hanson 2019)

Let κ *be a weakly compact cardinal and let* $\mathbb{P}$ *denote a length-*κ *finite-support iteration of Hechler forcings. Then*

$$V^{\mathbb{P}} \models \text{“}\lim^n \mathbf{A} = 0 \text{ for all } n > 0.\text{”}$$

Theorem (Bannister, B., Moore 2020)

In the model $V^{\mathbb{P}}$ *appearing above, strong homology is additive on the class of locally compact separable metric spaces.*

Theorem (B., Hrušák, Lambie-Hanson 2021)

It is consistent with the ZFC axioms that $2^{\aleph_0} = \aleph_{\omega+1}$ *and* $\lim^n \mathbf{A} = 0$ *for all* $n > 0$.

a factorization

a factorization

Let's note the outstanding remaining open question in this line:

Question

What is the minimum value of the continuum compatible with the assertion "$\lim^n \mathbf{A} = 0$ *for all* $n > 0$"?

a factorization

Let's note the outstanding remaining open question in this line:

Question

What is the minimum value of the continuum compatible with the assertion "$\lim^n \mathbf{A} = 0$ *for all* $n > 0$"*?*

Write Ω for the quasi-order $({}^\omega\omega, \leq^*)$.

a factorization

Let's note the outstanding remaining open question in this line:

Question

What is the minimum value of the continuum compatible with the assertion "$\lim^n \mathbf{A} = 0$ *for all* $n > 0$*"?*

Write Ω for the quasi-order $({}^\omega\omega, \leq^*)$. We turn our focus now to a family of purely set-theoretic principles $\mathrm{PH}_n(\Omega)$ $(n \in \omega)$ through which the above results *factor* in the following sense:

Theorem (Bannister, B., Moore, Todorcevic 2022)

Let $V^{\mathbb{P}}$ *be as above; then*

$$V^{\mathbb{P}} \models \text{“}\mathrm{PH}_n(\Omega) \text{ for all } n \in \omega.\text{”}$$

Theorem (Bannister, B., Moore, Todorcevic 2022)

"$\mathrm{PH}_n(\Omega)$ for all $n \in \omega$" *implies that strong homology is additive on the class of locally compact separable metric spaces.*

the principles in question

the principles in question

Notation (1)

Let Q be a quasi-order.

the principles in question

Notation (1)

Let Q *be a quasi-order. For any* $n \geq 1$ *let*

$$Q^{\leq n} = \bigcup_{i=1}^{n} Q^i$$

and for any $\sigma, \tau \in Q^{\leq n}$*, write* $\sigma \trianglelefteq \tau$ *if* σ *is a subsequence of* τ*.*

the principles in question

Notation (1)

Let Q *be a quasi-order. For any* $n \geq 1$ *let*

$$Q^{\leq n} = \bigcup_{i=1}^{n} Q^i$$

and for any $\sigma, \tau \in Q^{\leq n}$, *write* $\sigma \trianglelefteq \tau$ *if* σ *is a subsequence of* τ.

Let

$$Q^{[n]} = \{\sigma \in Q^n \mid i \leq j \Rightarrow \sigma(i) \leq \sigma(j)\}$$

and let

$$Q^{[[n]]} = \{\bar{\sigma} \in \prod_{i=1}^{n} Q^i \mid i \leq j \Rightarrow \bar{\sigma}(i) \trianglelefteq \bar{\sigma}(j)\}.$$

the principles in question

Notation (2)

For any $n \geq 1$*, say a function* $F : Q^{\leq n} \to Q$ *is* *n-cofinal* *if*

1. $s \leq F(s)$ *for all* $s \in Q$*, and*
2. $\sigma \trianglelefteq \tau \Rightarrow F(\sigma) \leq F(\tau)$.

Notation (2)

For any $n \geq 1$, *say a function* $F : Q^{\leq n} \to Q$ *is* *n-cofinal* *if*

1. $s \leq F(s)$ *for all* $s \in Q$, *and*
2. $\sigma \trianglelefteq \tau \Rightarrow F(\sigma) \leq F(\tau)$.

Observe that such a function induces an

$$F^* : Q^{[[n]]} \to Q^{[n]}$$

given by

$$F^*(\bar{\sigma}) = F^* \circ \sigma = \langle F(\bar{\sigma}(i)) \mid 1 \leq i \leq n \rangle.$$

Notation (2)

For any $n \geq 1$*, say a function* $F : Q^{\leq n} \to Q$ *is* *n-cofinal* *if*

1. $s \leq F(s)$ *for all* $s \in Q$*, and*
2. $\sigma \trianglelefteq \tau \Rightarrow F(\sigma) \leq F(\tau)$.

Observe that such a function induces an

$$F^* : Q^{[[n]]} \to Q^{[n]}$$

given by

$$F^*(\bar{\sigma}) = F^* \circ \sigma = \langle F(\bar{\sigma}(i)) \mid 1 \leq i \leq n \rangle.$$

Observe also that if $F : Q^{\leq n} \to Q$ is n-cofinal and $1 \leq m \leq n$ then $F \restriction Q^{\leq m}$ is m-cofinal.

the principles in question

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

Exercise

Is $\mathrm{PH}_0(\omega)$ *true or false?*

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

Exercise

Is $\mathrm{PH}_0(\omega)$ *true or false?*

Answer

False; hence $\mathrm{PH}_n(\omega)$ *is false for all* $n \in \omega$*.*

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

Exercise

Is $\mathrm{PH}_1(\omega_1)$ *true or false?*

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

Exercise

Is $\mathrm{PH}_1(\omega_1)$ *true or false?*

Answer

False again.

the principles in question

Definition

For any $n \geq 1$ the partition hypothesis associated to Q^{n+1} is:

> $\mathrm{PH}_n(Q)$: *For all* $c : Q^{n+1} \to \omega$ *there is an* $(n+1)$*-cofinal* $F : Q^{\leq n+1} \to Q$ *such that* $c \circ F^*$ *is constant.*

Note that by our observations on the preceding slide, these principles are monotonic in n; in other words,

$$\mathrm{PH}_n(Q) \Rightarrow \mathrm{PH}_m(Q) \text{ for all } 1 \leq m \leq n.$$

Exercise

Is $\mathrm{PH}_1(\omega_1)$ *true or false?*

Answer

False again. In fact...

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

$\mathrm{PH}_n(\omega_n)$ *is false for every* $n \in \omega$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

$\mathrm{PH}_n(\omega_n)$ *is false for every* $n \in \omega$.

Proof sketch.

The idea of the argument is this.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

$\mathrm{PH}_n(\omega_n)$ is false for every $n \in \omega$.

Proof sketch.

The idea of the argument is this. We inductively define colorings $c_n : (\omega_n)^{n+1} \to \omega$ which witness the failure of $\mathrm{PH}_n(\omega_n)$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

$\mathrm{PH}_n(\omega_n)$ *is false for every* $n \in \omega$.

Proof sketch.

The idea of the argument is this. We inductively define colorings $c_n : (\omega_n)^{n+1} \to \omega$ which witness the failure of $\mathrm{PH}_n(\omega_n)$. Each essentially derives from a coloring $\tilde{c}_n : [\omega_n]^{n+1} \to \omega$ defined as follows: for each $\gamma \in \omega_n$ fix an injection $f_\gamma : \gamma \hookrightarrow \omega_{n-1}$ and let

$$\tilde{c}_n(\vec{\alpha}, \gamma) = \tilde{c}_{n-1}(f_\gamma(\alpha_0), \ldots, f_\gamma(\alpha_{n-1}))$$

for any $\alpha_0 < \cdots < \alpha_{n-1} < \gamma < \omega_n$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

$\mathrm{PH}_n(\omega_n)$ *is false for every* $n \in \omega$.

Proof sketch.

The idea of the argument is this. We inductively define colorings $c_n : (\omega_n)^{n+1} \to \omega$ which witness the failure of $\mathrm{PH}_n(\omega_n)$. Each essentially derives from a coloring $\tilde{c}_n : [\omega_n]^{n+1} \to \omega$ defined as follows: for each $\gamma \in \omega_n$ fix an injection $f_\gamma : \gamma \hookrightarrow \omega_{n-1}$ and let

$$\tilde{c}_n(\vec{\alpha}, \gamma) = \tilde{c}_{n-1}(f_\gamma(\alpha_0), \dots, f_\gamma(\alpha_{n-1}))$$

for any $\alpha_0 < \cdots < \alpha_{n-1} < \gamma < \omega_n$. Proving that for any n-cofinal $F : (\omega_n)^{\leq n+1} \to \omega_n$ the function $c_n \circ F^*$ is nonconstant is where things get interesting: the indispensible tools, as it turned out, were simplicial homology, and a view of these accumulating $\vec{\alpha} * \gamma$ relations as simplicial cones. $\square$

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ *is a ZFC theorem for any* $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ *is a* ZFC *theorem for any* $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_0(\xi)$ holds, in particular, for $\xi = \omega_1$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ *is a* ZFC *theorem for any* $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_0(\xi)$ holds, in particular, for $\xi = \omega_1$.

Question

What about $\mathrm{PH}_1(\omega_2)$*?*

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ is a ZFC theorem for any $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_0(\xi)$ holds, in particular, for $\xi = \omega_1$.

Question

What about $\mathrm{PH}_1(\omega_2)$?

Theorem

If there exists a uniform, countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2, then $\mathrm{PH}_1(\omega_2)$ holds.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ is a ZFC theorem for any $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_0(\xi)$ holds, in particular, for $\xi = \omega_1$.

Question

What about $\mathrm{PH}_1(\omega_2)$?

Theorem

If there exists a uniform, countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2, then $\mathrm{PH}_1(\omega_2)$ holds.

Recall an ideal $\mathcal{I}$ is $\aleph_1$-dense if $P(\omega_2)/\mathcal{I}$ contains a dense set of size $\aleph_1$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

On the other hand:

Lemma

$\mathrm{PH}_0(\xi)$ is a ZFC theorem for any $\xi \notin \mathrm{Cof}(\aleph_0)$.

$\mathrm{PH}_0(\xi)$ holds, in particular, for $\xi = \omega_1$.

Question

What about $\mathrm{PH}_1(\omega_2)$?

Theorem

If there exists a uniform, countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2, then $\mathrm{PH}_1(\omega_2)$ holds.

Recall an ideal $\mathcal{I}$ is $\aleph_1$-dense if $P(\omega_2)/\mathcal{I}$ contains a dense set of size $\aleph_1$. Recall also that Foreman has constructed such ideals, beginning from the assumption of a huge cardinal.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow$ $\mathrm{PH}_1(\omega_2)$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant. First, fix for each $\alpha \in \omega_2$ an $A_\alpha \in \mathcal{I}^+$ and $i_\alpha \in \omega$ such that $c(\alpha, \beta) = i_\alpha$ for all $\beta \in A_\alpha$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant. First, fix for each $\alpha \in \omega_2$ an $A_\alpha \in \mathcal{I}^+$ and $i_\alpha \in \omega$ such that $c(\alpha, \beta) = i_\alpha$ for all $\beta \in A_\alpha$. Since $\mathcal{I}$ is $\aleph_1$-dense, there exist $B \in [\omega_2]^{\omega_2}$ and $X \in \mathcal{I}^+$ such that $A_\alpha \supseteq X \pmod{\mathcal{I}}$ for all $\alpha \in B$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant. First, fix for each $\alpha \in \omega_2$ an $A_\alpha \in \mathcal{I}^+$ and $i_\alpha \in \omega$ such that $c(\alpha, \beta) = i_\alpha$ for all $\beta \in A_\alpha$. Since $\mathcal{I}$ is $\aleph_1$-dense, there exist $B \in [\omega_2]^{\omega_2}$ and $X \in \mathcal{I}^+$ such that $A_\alpha \supseteq X \pmod{\mathcal{I}}$ for all $\alpha \in B$. Fix $i \in \omega$ and $C \in [B]^{\omega_2}$ with $i_\alpha = i$ for all $\alpha \in C$.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant. First, fix for each $\alpha \in \omega_2$ an $A_\alpha \in \mathcal{I}^+$ and $i_\alpha \in \omega$ such that $c(\alpha, \beta) = i_\alpha$ for all $\beta \in A_\alpha$. Since $\mathcal{I}$ is $\aleph_1$-dense, there exist $B \in [\omega_2]^{\omega_2}$ and $X \in \mathcal{I}^+$ such that $A_\alpha \supseteq X \pmod{\mathcal{I}}$ for all $\alpha \in B$. Fix $i \in \omega$ and $C \in [B]^{\omega_2}$ with $i_\alpha = i$ for all $\alpha \in C$. For each $\alpha \in \omega_2$ let $F(\alpha) = \min C \setminus \alpha$ and for $\alpha \le \beta$ in ω_2 let

$$F(\alpha, \beta) = \min(A_{F(\alpha)} \cap A_{F(\beta)} \setminus F(\alpha) \cup F(\beta)).$$

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

Theorem

A countably complete, $\aleph_1$-dense ideal $\mathcal{I}$ on ω_2 $\Rightarrow \mathrm{PH}_1(\omega_2)$.

Proof.

Fix a $c : (\omega_2)^2 \to \omega$ and $\mathcal{I}$ as above. We will define a 2-cofinal F with $c \circ F^*$ constant. First, fix for each $\alpha \in \omega_2$ an $A_\alpha \in \mathcal{I}^+$ and $i_\alpha \in \omega$ such that $c(\alpha, \beta) = i_\alpha$ for all $\beta \in A_\alpha$. Since $\mathcal{I}$ is $\aleph_1$-dense, there exist $B \in [\omega_2]^{\omega_2}$ and $X \in \mathcal{I}^+$ such that $A_\alpha \supseteq X \pmod{\mathcal{I}}$ for all $\alpha \in B$. Fix $i \in \omega$ and $C \in [B]^{\omega_2}$ with $i_\alpha = i$ for all $\alpha \in C$. For each $\alpha \in \omega_2$ let $F(\alpha) = \min C \backslash \alpha$ and for $\alpha \leq \beta$ in ω_2 let

$$F(\alpha, \beta) = \min(A_{F(\alpha)} \cap A_{F(\beta)} \backslash F(\alpha) \cup F(\beta)).$$

Since both $A_{F(\alpha)}$ and $A_{F(\beta)}$ contain $X \pmod{\mathcal{I}}$, this expression is well-defined. $\square$

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

It's clear that $\mathrm{PH}_1(\omega_2)$ entails *some* large cardinal assumptions:

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

It's clear that $\mathrm{PH}_1(\omega_2)$ entails *some* large cardinal assumptions:

Theorem

For all $n > 0$, if κ is regular and $\mathrm{PH}_n(\kappa)$ holds then κ is weakly compact in L.

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

It's clear that $\mathrm{PH}_1(\omega_2)$ entails *some* large cardinal assumptions:

Theorem

For all $n > 0$, if κ is regular and $\mathrm{PH}_n(\kappa)$ holds then κ is weakly compact in L.

Question

What is the consistency strength of $\mathrm{PH}_1(\omega_2)$?

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

It's clear that $\mathrm{PH}_1(\omega_2)$ entails *some* large cardinal assumptions:

Theorem

For all $n > 0$, if κ is regular and $\mathrm{PH}_n(\kappa)$ holds then κ is weakly compact in L.

Question

What is the consistency strength of $\mathrm{PH}_1(\omega_2)$? Or of $\mathrm{PH}_2(\omega_3)$?

$\mathrm{PH}_n(\xi)$ for ξ an ordinal

It's clear that $\mathrm{PH}_1(\omega_2)$ entails *some* large cardinal assumptions:

Theorem

For all $n > 0$, if κ is regular and $\mathrm{PH}_n(\kappa)$ holds then κ is weakly compact in L.

Question

What is the consistency strength of $\mathrm{PH}_1(\omega_2)$? Or of $\mathrm{PH}_2(\omega_3)$?

Theorem

If there exist uniform ideals $\mathcal{I}$ and $\mathcal{J}$ on a cardinal κ satisfying

- $\aleph_1 \leq \mathrm{add}(\mathcal{I})$,
- $\mathrm{dens}(\mathcal{I}^+) < \mathrm{add}(\mathcal{J}) \leq \kappa$, *and*
- $\mathrm{dens}(\mathcal{J}^+) < \kappa$,

then $\mathrm{PH}_2(\kappa)$ holds.

transmitting PH_n

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

Lemma

If P and Q are directed quasi-orders and $P \to Q$ is a monotone map with cofinal image, then $\mathrm{PH}_n(P)$ implies $\mathrm{PH}_n(Q)$.

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

Lemma

If P and Q are directed quasi-orders and $P \to Q$ is a monotone map with cofinal image, then $\mathrm{PH}_n(P)$ implies $\mathrm{PH}_n(Q)$.

In particular, $\mathrm{PH}_n(\xi)$ is equivalent to $\mathrm{PH}_n(\mathrm{cf}(\xi))$.

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

Lemma

If P and Q are directed quasi-orders and $P \to Q$ is a monotone map with cofinal image, then $\mathrm{PH}_n(P)$ implies $\mathrm{PH}_n(Q)$.

In particular, $\mathrm{PH}_n(\xi)$ is equivalent to $\mathrm{PH}_n(\mathrm{cf}(\xi))$. Hence:

Theorem

The partition hypothesis $\mathrm{PH}_n(\Omega)$ implies $\mathrm{PH}_n(\mathfrak{d})$. In particular, it implies that $\mathrm{cf}(\mathfrak{d}) > \omega_n$, and if $n > 0$, it further implies that $\mathrm{cf}(\mathfrak{d})$ is weakly compact in L.

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

Lemma

If P and Q are directed quasi-orders and $P \to Q$ is a monotone map with cofinal image, then $\mathrm{PH}_n(P)$ implies $\mathrm{PH}_n(Q)$.

In particular, $\mathrm{PH}_n(\xi)$ is equivalent to $\mathrm{PH}_n(\mathrm{cf}(\xi))$. Hence:

Theorem

The partition hypothesis $\mathrm{PH}_n(\Omega)$ implies $\mathrm{PH}_n(\mathfrak{d})$. In particular, it implies that $\mathrm{cf}(\mathfrak{d}) > \omega_n$, and if $n > 0$, it further implies that $\mathrm{cf}(\mathfrak{d})$ is weakly compact in L. Hence for all $n > 0$ the consistency strength of the principle $\mathrm{PH}_n(\Omega)$ is exactly a weakly compact cardinal.

transmitting PH_n

It's worth observing that PH_n isn't just about the cardinal κ.

Lemma

If P and Q are directed quasi-orders and $P \to Q$ is a monotone map with cofinal image, then $\mathrm{PH}_n(P)$ implies $\mathrm{PH}_n(Q)$.

In particular, $\mathrm{PH}_n(\xi)$ is equivalent to $\mathrm{PH}_n(\mathrm{cf}(\xi))$. Hence:

Theorem

The partition hypothesis $\mathrm{PH}_n(\Omega)$ implies $\mathrm{PH}_n(\mathfrak{d})$. In particular, it implies that $\mathrm{cf}(\mathfrak{d}) > \omega_n$, and if $n > 0$, it further implies that $\mathrm{cf}(\mathfrak{d})$ is weakly compact in L. Hence for all $n > 0$ the consistency strength of the principle $\mathrm{PH}_n(\Omega)$ is exactly a weakly compact cardinal.

Proof idea.

If $\langle f_\alpha \mid \alpha < \mathfrak{d} \rangle$ is $\leq^*$-cofinal in ${}^\omega\omega$, then $f \mapsto \min\{\alpha \mid f \leq^* f_\alpha\}$ is a monotone map $\Omega \to \mathfrak{d}$ with cofinal image. $\square$

a seeming digression

a seeming digression

A persistent recognition in the study of topology is of homology theory, and even more particularly homotopy theory, as fundamentally combinatorial in nature.

a seeming digression

A persistent recognition in the study of topology is of homology theory, and even more particularly homotopy theory, as fundamentally combinatorial in nature. A watershed (1967) formalization of this recognition is the *Quillen equivalence* of the standard *homotopy theoretic*, or more precisely, *model category* structures on the category of topological spaces and the category of simplicial sets.

a seeming digression

A persistent recognition in the study of topology is of homology theory, and even more particularly homotopy theory, as fundamentally combinatorial in nature. A watershed (1967) formalization of this recognition is the *Quillen equivalence* of the standard *homotopy theoretic*, or more precisely, *model category* structures on the category of topological spaces and the category of simplicial sets. Please bear with me: I want to talk a little while about the latter.

simplicial sets

simplicial sets

Definition

Write Δ for the category of finite nonempty ordinals, whose objects are (in these contexts) typically written $[0] = \{0\}$, $[1] = \{0, 1\}$, $[n] = \{0, \ldots, n\}$, etc., and whose morphisms are order-preserving maps $f : [m] \to [n]$.

simplicial sets

Definition

Write Δ for the category of finite nonempty ordinals, whose objects are (in these contexts) typically written $[0] = \{0\}$, $[1] = \{0, 1\}$, $[n] = \{0, \ldots, n\}$, etc., and whose morphisms are order-preserving maps $f : [m] \to [n]$. A simplicial set X is a contravariant functor $S : \Delta \to \mathsf{Set}$; put differently, it is a collection of sets $S([n]) = X_n$ (whose elements are thought of as the *n-dimensional faces*, or *n-simplices*, of X), for $n \in \omega$,

simplicial sets

Definition

Write Δ for the category of finite nonempty ordinals, whose objects are (in these contexts) typically written $[0] = \{0\}$, $[1] = \{0, 1\}$, $[n] = \{0, \ldots, n\}$, etc., and whose morphisms are order-preserving maps $f : [m] \to [n]$. A simplicial set X is a contravariant functor $S : \Delta \to \mathsf{Set}$; put differently, it is a collection of sets $S([n]) = X_n$ (whose elements are thought of as the *n-dimensional faces*, or *n-simplices*, of X), for $n \in \omega$, together with morphisms $S(f) : X_n \to X_m$ (thought of as *face maps*) whose relations mirror those among the morphisms in Δ.

ordered simplicial complexes

ordered simplicial complexes

Example

Let Y be your favorite ordered (abstract) simplicial complex — i.e., Y is a totally ordered set of vertices V, together with a downwards closed collection of finite subsets σ of V.

ordered simplicial complexes

Example

Let Y be your favorite ordered (abstract) simplicial complex — i.e., Y is a totally ordered set of vertices V, together with a downwards closed collection of finite subsets σ of V. Letting $X_n = \{\sigma \in Y \mid |\sigma| = n+1\}$ *almost* determines a simplicial set; what's missing are the "degenerate" faces of Y, those of the form $(1,3,3,5)$ (if $(1,3,5) \in Y$), etc.

ordered simplicial complexes

Example

Let Y be your favorite ordered (abstract) simplicial complex — i.e., Y is a totally ordered set of vertices V, together with a downwards closed collection of finite subsets σ of V. Letting $X_n = \{\sigma \in Y \mid |\sigma| = n+1\}$ *almost* determines a simplicial set; what's missing are the "degenerate" faces of Y, those of the form $(1,3,3,5)$ (if $(1,3,5) \in Y$), etc. Adding these in defines the simplicial set X associated to Y; the structuring maps in X are generated by face and coface maps I'll probably prefer to describe on the board...

ordered simplicial complexes

Example

Let Y be your favorite ordered (abstract) simplicial complex — i.e., Y is a totally ordered set of vertices V, together with a downwards closed collection of finite subsets σ of V. Letting $X_n = \{\sigma \in Y \mid |\sigma| = n+1\}$ *almost* determines a simplicial set; what's missing are the "degenerate" faces of Y, those of the form $(1,3,3,5)$ (if $(1,3,5) \in Y$), etc. Adding these in defines the simplicial set X associated to Y; the structuring maps in X are generated by face and coface maps I'll probably prefer to describe on the board... Let's recall also, while I'm there, the notion of the *subdivision* of an abstract simplicial complex.

realization and homotopy

realization and homotopy

Like simplicial complexes, simplicial sets admit geometric realizations — in fact there exists a geometric realization functor $T : \mathsf{sSet} \to \mathsf{Top}$ which is left adjoint to the *singular* functor $S : \mathsf{Top} \to \mathsf{sSet}$, i.e.,

$$\mathrm{Hom}_{\mathsf{Top}}(TX, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, SY)$$

for any simplicial set X and topological space Y.

realization and homotopy

Like simplicial complexes, simplicial sets admit geometric realizations — in fact there exists a geometric realization functor $T : \mathsf{sSet} \to \mathsf{Top}$ which is left adjoint to the *singular* functor $S : \mathsf{Top} \to \mathsf{sSet}$, i.e.,

$$\mathrm{Hom}_{\mathsf{Top}}(TX, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, SY)$$

for any simplicial set X and topological space Y. ($(SY)_n$ is just the set of continuous maps from an n-simplex to the topological space Y; it should be clear then how maps $[m] \to [n]$ induce functions $(SY)_n \to (SY)_m$. S is, of course, the functor underlying the singular homology of Y).

realization and homotopy

Like simplicial complexes, simplicial sets admit geometric realizations — in fact there exists a geometric realization functor $T : \mathsf{sSet} \to \mathsf{Top}$ which is left adjoint to the *singular* functor $S : \mathsf{Top} \to \mathsf{sSet}$, i.e.,

$$\mathrm{Hom}_{\mathsf{Top}}(TX, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, SY)$$

for any simplicial set X and topological space Y. ($(SY)_n$ is just the set of continuous maps from an n-simplex to the topological space Y; it should be clear then how maps $[m] \to [n]$ induce functions $(SY)_n \to (SY)_m$. S is, of course, the functor underlying the singular homology of Y). The remarkable point is that whatever distortions or identifications these functors might introduce, they *do* respect the natural notions of homotopy (fibrations, cofibrations, weak equivalences) on each side.

realization and homotopy

Like simplicial complexes, simplicial sets admit geometric realizations — in fact there exists a geometric realization functor $T : \mathsf{sSet} \to \mathsf{Top}$ which is left adjoint to the *singular* functor $S : \mathsf{Top} \to \mathsf{sSet}$, i.e.,

$$\mathrm{Hom}_{\mathsf{Top}}(TX, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, SY)$$

for any simplicial set X and topological space Y. ($(SY)_n$ is just the set of continuous maps from an n-simplex to the topological space Y; it should be clear then how maps $[m] \to [n]$ induce functions $(SY)_n \to (SY)_m$. S is, of course, the functor underlying the singular homology of Y). The remarkable point is that whatever distortions or identifications these functors might introduce, they *do* respect the natural notions of homotopy (fibrations, cofibrations, weak equivalences) on each side. It's this fact which the phrase *Quillen equivalence* should be understood to primarily signify.

realization and homotopy

Like simplicial complexes, simplicial sets admit geometric realizations — in fact there exists a geometric realization functor $T : \mathsf{sSet} \to \mathsf{Top}$ which is left adjoint to the *singular* functor $S : \mathsf{Top} \to \mathsf{sSet}$, i.e.,

$$\mathrm{Hom}_{\mathsf{Top}}(TX, Y) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, SY)$$

for any simplicial set X and topological space Y. ($(SY)_n$ is just the set of continuous maps from an n-simplex to the topological space Y; it should be clear then how maps $[m] \to [n]$ induce functions $(SY)_n \to (SY)_m$. S is, of course, the functor underlying the singular homology of Y). The remarkable point is that whatever distortions or identifications these functors might introduce, they *do* respect the natural notions of homotopy (fibrations, cofibrations, weak equivalences) on each side. It's this fact which the phrase *Quillen equivalence* should be understood to primarily signify. Moreover, each category contains a class of objects best suited for homotopy operations: the *CW complexes* in Top; the *Kan complexes* in sSet.

I should probably come to the point

I should probably come to the point

We return to the orbit of what this talk is supposed to be about with one more example of a simplicial set.

I should probably come to the point

We return to the orbit of what this talk is supposed to be about with one more example of a simplicial set.

Example

For any quasi-order Q, let NQ denote the nerve of Q; this is the simplicial set whose n-simplices consist of the length-$(n+1)$ chains in Q.

I should probably come to the point

We return to the orbit of what this talk is supposed to be about with one more example of a simplicial set.

Example

For any quasi-order Q, let NQ denote the nerve of Q; this is the simplicial set whose n-simplices consist of the length-$(n+1)$ chains in Q.

In this perspective, a function $c : [\kappa]^{n+1} \to \lambda$, for example, is just a coloring of the nondegenerate n-faces of $N\kappa$.

I should probably come to the point

We return to the orbit of what this talk is supposed to be about with one more example of a simplicial set.

Example

For any quasi-order Q, let NQ denote the nerve of Q; this is the simplicial set whose n-simplices consist of the length-$(n+1)$ chains in Q.

In this perspective, a function $c : [\kappa]^{n+1} \to \lambda$, for example, is just a coloring of the nondegenerate n-faces of $N\kappa$. It's useful to introduce some further notation at this stage.

I should probably come to the point

We return to the orbit of what this talk is supposed to be about with one more example of a simplicial set.

Example

For any quasi-order Q, let NQ denote the nerve of Q; this is the simplicial set whose n-simplices consist of the length-$(n+1)$ chains in Q.

In this perspective, a function $c : [\kappa]^{n+1} \to \lambda$, for example, is just a coloring of the nondegenerate n-faces of $N\kappa$. It's useful to introduce some further notation at this stage.

Notation

For any $n \geq 0$ the family of maps $v_i : [0] \to [n] : 0 \mapsto i$ determines a family of maps $v_i^* : X_n \to X_0$. Let

$$\mathrm{vert}(x) = \{v_i^*(x) \mid i \in [n]\}$$

for any $x \in X_n$.

nerves, Kan complexes, and Ex

nerves, Kan complexes, and Ex

The nerve functor N fully and faithfully embeds the category of quasi-orders into the category of simplicial sets.

nerves, Kan complexes, and *Ex*

The nerve functor N fully and faithfully embeds the category of quasi-orders into the category of simplicial sets. Within the framework of the latter, N has one main shortcoming, which is that the nerve of a (nontrivial) quasi-order is never a *Kan complex*.

nerves, Kan complexes, and *Ex*

The nerve functor N fully and faithfully embeds the category of quasi-orders into the category of simplicial sets. Within the framework of the latter, N has one main shortcoming, which is that the nerve of a (nontrivial) quasi-order is never a *Kan complex*. More important than Kan complexes' definition for us right now is their analogy with CW complexes: just as any topological space may be replaced by a weakly equivalent CW complex, any simplicial set may be converted to a Kan complex via repeated applications of the Ex functor, which should be thought of as the *reverse* (or more precisely right adjoint) *of the subdivision functor on simplicial sets*.

nerves, Kan complexes, and Ex

The nerve functor N fully and faithfully embeds the category of quasi-orders into the category of simplicial sets. Within the framework of the latter, N has one main shortcoming, which is that the nerve of a (nontrivial) quasi-order is never a *Kan complex*. More important than Kan complexes' definition for us right now is their analogy with CW complexes: just as any topological space may be replaced by a weakly equivalent CW complex, any simplicial set may be converted to a Kan complex via repeated applications of the Ex functor, which should be thought of as the *reverse* (or more precisely right adjoint) *of the subdivision functor on simplicial sets*.

Soft Definition

For any quasi-order Q*, the elements of* $(Ex\, NQ)_n$ *are* the copies of the subdivision of the standard abstract n-simplex in Q.

PH_n restated

PH_n restated

To come now fully to the point:

Definition

For any simplicial set X and $n > 0$, say $Z \subseteq X_n$ *spans* $Y \subseteq X_0$ if for every $\bar{y} \in [Y]^{n+1}$ there exists a $z \in Z$ with $\mathrm{vert}(Z) = \bar{y}$.

PH_n restated

To come now fully to the point:

Definition

For any simplicial set X and $n > 0$, say $Z \subseteq X_n$ *spans* $Y \subseteq X_0$ if for every $\bar{y} \in [Y]^{n+1}$ there exists a $z \in Z$ with $\mathrm{vert}(Z) = \bar{y}$.

$\kappa \to (\kappa)^{n+1}_{\omega}$ then translates as:

For all $c : [\kappa]^{n+1} \to \omega$ there exists a cofinal $Y \subseteq \kappa$ and c-monochromatic $Z \subseteq (N\kappa)_n$ spanning Y.

And $\mathrm{PH}_n(\kappa)$ translates as:

For all $c : \kappa^{n+1} \to \omega$ there exists a cofinal $Y \subseteq \kappa$ and c-monochromatic $Z \subseteq (Ex\, N\kappa)_n$ spanning Y.

More generally, $\mathrm{PH}_n(Q)$ asserts for any quasi-order Q that:

For all $c : Q^{n+1} \to \omega$ there exists a cofinal $Y \subseteq Q$ and c-monochromatic $Z \subseteq (Ex\, NQ)_n$ spanning Y.

PH_n for definable partitions

PH_n for definable partitions

Just to recap: at the heart of several recent results on the vanishing of $\lim^n$, the arguments of which had all involved a distracting amount of algebra, is the *purely combinatorial principle* PH_n.

PH_n for definable partitions

Just to recap: at the heart of several recent results on the vanishing of $\lim^n$, the arguments of which had all involved a distracting amount of algebra, is the *purely combinatorial principle* PH_n. This decomposition of those arguments facilitates a closer analysis of their *descriptive set theoretic content*, and this is what motivated the study of these principles in the first place.

PH_n for definable partitions

Just to recap: at the heart of several recent results on the vanishing of $\lim^n$, the arguments of which had all involved a distracting amount of algebra, is the *purely combinatorial principle* PH_n. This decomposition of those arguments facilitates a closer analysis of their *descriptive set theoretic content*, and this is what motivated the study of these principles in the first place. More particularly, we were interested in the higher-n versions of Todorcevic's aforementioned result that *any analytic coherent family* $\Phi = \{\varphi_f : \ell(f) \to \mathbb{Z} \mid f \in {}^\omega\omega\}$ *is trivial.*

universally Baire n-coherent families

universally Baire n-coherent families

Under a hypothesis which we denote (†) (with consistency strength an inaccessible cardinal), Todorcevic's result admits the following strong generalization:

universally Baire n-coherent families

Under a hypothesis which we denote (†) (with consistency strength an inaccessible cardinal), Todorcevic's result admits the following strong generalization:

Theorem (†)

Any universally Baire n-coherent family $\Phi = \{\varphi_{\vec{f}} \mid \vec{f} \in ({}^\omega\omega)^n\}$ admits a Σ^2_1 trivialization.

(Recall that a subset A of a Polish space Y is *universally Baire* if for any topological space X and continuous $f : X \to Y$, $f^{-1}(A)$ has the property of Baire in X. (And recall that a subset B of a topological space X has *the property of Baire* if there is an open $U \subseteq X$ such that the symmetric difference of B and U is meager in X.))

universally Baire n-coherent families

Under a hypothesis which we denote (†) (with consistency strength an inaccessible cardinal), Todorcevic's result admits the following strong generalization:

Theorem (†)

Any universally Baire n-coherent family $\Phi = \{\varphi_{\vec{f}} \mid \vec{f} \in ({}^{\omega}\omega)^n\}$ admits a Σ^2_1 trivialization.

(Recall that a subset A of a Polish space Y is *universally Baire* if for any topological space X and continuous $f : X \to Y$, $f^{-1}(A)$ has the property of Baire in X. (And recall that a subset B of a topological space X has *the property of Baire* if there is an open $U \subseteq X$ such that the symmetric difference of B and U is meager in X.)) Relatedly:

Theorem

Suppose that there is a supercompact cardinal or a proper class of Woodin cardinals. Then $L(\mathbb{R}) \models$ "$\lim^n \mathbf{A} = 0$ *for all* $n > 0$".

$\mathcal{H}_n$-measurability and PH_n

$\mathcal{H}_n$-measurability and PH_n

Back of these results is an analysis of $\mathrm{PH}_n(\Omega)$ in relation to notions of $\mathcal{H}_n$-*meagerness* and $\mathcal{H}_n$-*measurability* which are closely tied to n-fold iterations of Hechler forcing.

Definition

Let *the Hechler topology* τ denote the topology on ${}^\omega\omega$ which is generated by the basic open sets

$$N_k(f) := \{g \in {}^\omega\omega \mid g \geq f \text{ and } g \restriction k = f \restriction k\}.$$

$\mathcal{H}_n$-measurability and PH_n

Back of these results is an analysis of $\mathrm{PH}_n(\Omega)$ in relation to notions of $\mathcal{H}_n$-*meagerness* and $\mathcal{H}_n$-*measurability* which are closely tied to n-fold iterations of Hechler forcing.

Definition

Let *the Hechler topology* τ denote the topology on ${}^\omega\omega$ which is generated by the basic open sets

$$N_k(f) := \{g \in {}^\omega\omega \mid g \geq f \text{ and } g \upharpoonright k = f \upharpoonright k\}.$$

Note that this topology is first countable and Choquet (a strong way of saying that $({}^\omega\omega, \tau)$ is a Baire space).

$\mathcal{H}_n$-measurability and PH_n

Back of these results is an analysis of $\mathrm{PH}_n(\Omega)$ in relation to notions of $\mathcal{H}_n$-*meagerness* and $\mathcal{H}_n$-*measurability* which are closely tied to n-fold iterations of Hechler forcing.

Definition

Let *the Hechler topology* τ denote the topology on ${}^\omega\omega$ which is generated by the basic open sets

$$N_k(f) := \{g \in {}^\omega\omega \mid g \geq f \text{ and } g \restriction k = f \restriction k\}.$$

Note that this topology is first countable and Choquet (a strong way of saying that $({}^\omega\omega, \tau)$ is a Baire space).

Definition

$X \subseteq {}^\omega\omega$ is $\mathcal{H}_1$-*meager* if it is a countable union of τ-nowhere dense sets. X is $\mathcal{H}_1$-*measurable* if it has the property of Baire with respect to τ.

$\mathcal{H}_n$-measurability and PH_n

Back of these results is an analysis of $\mathrm{PH}_n(\Omega)$ in relation to notions of $\mathcal{H}_n$-*meagerness* and $\mathcal{H}_n$-*measurability* which are closely tied to n-fold iterations of Hechler forcing.

Definition

Let *the Hechler topology* τ denote the topology on ${}^\omega\omega$ which is generated by the basic open sets

$$N_k(f) := \{g \in {}^\omega\omega \mid g \geq f \text{ and } g \upharpoonright k = f \upharpoonright k\}.$$

Note that this topology is first countable and Choquet (a strong way of saying that $({}^\omega\omega, \tau)$ is a Baire space).

Definition

$X \subseteq {}^\omega\omega$ is $\mathcal{H}_1$-*meager* if it is a countable union of τ-nowhere dense sets. X is $\mathcal{H}_1$-*measurable* if it has the property of Baire with respect to τ. The higher-n variants of these notions derive from $\mathcal{H}_1$ via a fairly complicated recursion.

Putting it all together

Putting it all together

We may now, at long last, state our key theorem. It is the following:

Putting it all together

We may now, at long last, state our key theorem. It is the following:

Theorem (†)

"$\mathrm{PH}_n(\Omega)$ *holds for measurable partitions*". *Somewhat more precisely, if* $c : ({}^{\omega}\omega)^{n+1} \to \omega$ *is an* $\mathcal{H}_{n+1}$*-measurable function then there exists an* $(n+1)$*-cofinal function* $F : \Omega^{\leq n+1} \to \Omega$ *which is* Σ^1_2 *and such that* $c \circ F^*$ *is constant.*

Putting it all together

We may now, at long last, state our key theorem. It is the following:

Theorem (†)

"$\mathrm{PH}_n(\Omega)$ holds for measurable partitions". Somewhat more precisely, if $c : ({}^{\omega}\omega)^{n+1} \to \omega$ is an $\mathcal{H}_{n+1}$-measurable function then there exists an $(n+1)$-cofinal function $F : \Omega^{\leq n+1} \to \Omega$ which is Σ^1_2 and such that $c \circ F^*$ is constant.

For full disclosure, the principle † should perhaps be recorded: it is the hypothesis that for every $n > 0$, every Σ^1_2 subset of $\Omega^{[n]}$ is $\mathcal{H}_n$-measurable. It holds in the Solovay model, and carries the consequence that every universally Baire Σ^1_2 subset of $\Omega^{[n]}$ is $\mathcal{H}_n$-measurable.

thanks

Many thanks to the organizers for the invitation,

and to the audience for their attention,

and for any questions which any of you may have.

thanks

Many thanks to the organizers for the invitation,

and to the audience for their attention,

and for any questions which any of you may have.