Cardinal characteristics and finite support iteration

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Forcing notions $\mathbb{P}, \mathbb{Q}, \dots$

Names: $V^{\mathbb{P}}$ $\quad \underset{\sim}{x} \in V^{\mathbb{P}} \iff \underset{\sim}{x} \subseteq \mathbb{P} \times V^{\mathbb{P}}$

or $V^{\mathbb{P}} \times \mathbb{P}$?

Evaluation: $\underset{\sim}{x}^G = \underset{\sim}{x}[G] = \{ \underset{\sim}{y}[G] : \exists p \in G : (p, \underset{\sim}{y}) \in \underset{\sim}{x} \}$

Forcing thm: $V \models p \Vdash \varphi(\underset{\sim}{x}) \iff \forall G \ni p : V[G] \models \varphi(\underset{\sim}{x}[G])$

(V- generic)

Composition: If $\mathbb{P} \Vdash \underset{\sim}{\mathbb{Q}}$ is a forcing notion, then

$$\mathbb{P} * \underset{\sim}{\mathbb{Q}} := \{ (p, \underset{\sim}{q}) \in \mathbb{P} \times \underset{\sim}{\mathbb{Q}} : p \Vdash \underset{\sim}{q} \in \underset{\sim}{\mathbb{Q}} \} \cap H(\chi)$$

Identify $\mathbb{P}$ with $\mathbb{P} \times \{1_{\mathbb{Q}}\} \lessdot \mathbb{P} * \mathbb{Q}$

General maxim:

adding desired objects is easy

avoiding undesired objects is ~~hard~~
needs work

add many reals

avoid collapsing cardinals

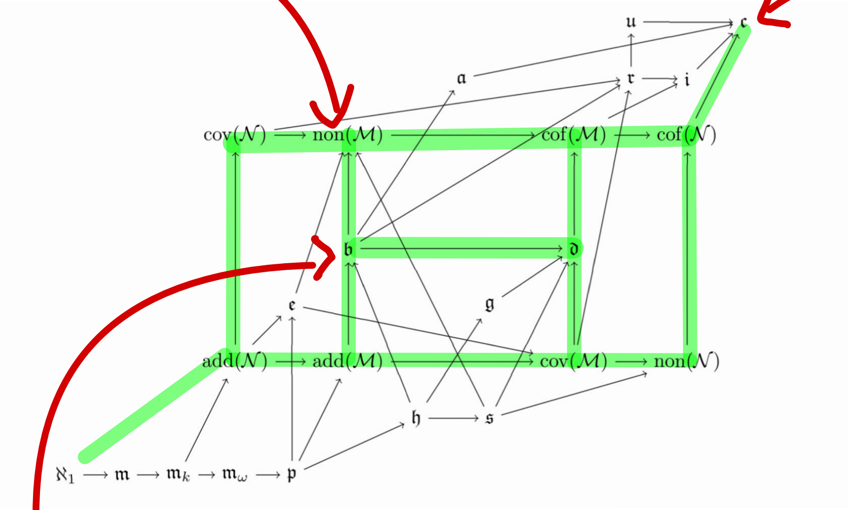
Cardinal characteristics of the continuum

(Augmented) Cichoń's Diagram:

$$\begin{tikzcd}
 & & & & & & & & & & \mathfrak{u} \arrow[rr] & & \mathfrak{c} \\
 & & & & & & & \mathfrak{a} \arrow[urrrrr] & & & \mathfrak{r} \arrow[u] \arrow[r] & \mathfrak{i} \arrow[ur] & \\
 & & & \mathrm{cov}(\mathcal{N}) \arrow[rr] \arrow[urrrrrrr] & & \mathrm{non}(\mathcal{M}) \arrow[rrrr] & & & & \mathrm{cof}(\mathcal{M}) \arrow[rr] \arrow[urr] & & \mathrm{cof}(\mathcal{N}) \arrow[uur] & \\
 & & & & & \mathfrak{b} \arrow[u] \arrow[rrrr] \arrow[uurr] \arrow[uurrrrr] & & & & \mathfrak{d} \arrow[u] & & & \\
 & & & & \mathfrak{e} \arrow[uur] \arrow[drrrrr] & & & & \mathfrak{g} \arrow[ur] & & & & \\
 & & & \mathrm{add}(\mathcal{N}) \arrow[rr] \arrow[uuu] \arrow[ur] & & \mathrm{add}(\mathcal{M}) \arrow[rrrr] \arrow[uu] & & & & \mathrm{cov}(\mathcal{M}) \arrow[rr] \arrow[uu] \arrow[uuuur] & & \mathrm{non}(\mathcal{N}) \arrow[uuu] & \\
 & & & & & & \mathfrak{h} \arrow[rr] \arrow[uuul] \arrow[uurr] & & \mathfrak{s} \arrow[uuuulll] \arrow[uuur] \arrow[urrr] & & & & \\
\aleph_1 \arrow[r] & \mathfrak{m} \arrow[r] & \mathfrak{m}_k \arrow[r] \arrow[uur] & \mathfrak{m}_\omega \arrow[r] & \mathfrak{p} \arrow[uuu] \arrow[uur] \arrow[urr] & & & & & & & &
\end{tikzcd}$$

non($\mathcal{M}$)= How many reals do you need so that no meager set contains all of them

$c = 2^{\aleph_0}$



How many reals ?

$(2^\omega, \mathbb{R}, \omega^\omega, \dots)$

Special meager sets $\subseteq \omega^\omega$:

$$M_g = \{f \mid \forall^\infty n: f(n) \neq g(n)\}$$

$$f \, R_{\text{nonM}} \, g \iff \forall^\infty n: f(n) \neq g(n)$$

$\mathfrak{b}$ = How many functions $f: \omega \to \omega$ do you need so that no $g: \omega \to \omega$ dominates all of them?

$$f \leq^* g \iff \overset{\forall^\infty n}{\exists n_0 \forall n \geq n_0} \; f(n) \leq g(n)$$

$$f \, R_{\mathfrak{b}} \, g$$

More general cardinal characteristics:

Let $X, Y \in \{2^\omega, \omega^\omega\}$. We consider binary relations $R \subseteq X \times Y$ with the following properties:

$$R = \bigcup_k R(k)$$

Each $R(k)$ closed.

$$\forall g: \{f \mid f\, R(k)\, g\} \text{ meager}$$

$$\forall f\, \exists g\; f R g$$

Def

$$b_R = \min\{|B| : B \subseteq X, \forall y \in Y : B \not R y\}$$

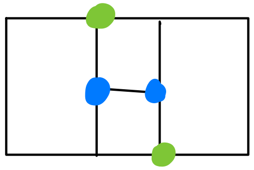
$\exists x \in B : x \not R y$

$$d_R = \min\{|D| : D \subseteq Y, \forall x \in X : x R D\}$$

$\exists y \in D \; x R y$

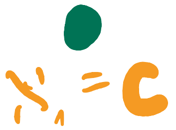
(Note: $d_R = b_{\neg \reflectbox{R}}$, $b_R = d_{\neg \reflectbox{R}}$, but $\neg R$ is usually not F_σ)

Examples: $f \leq^* g$ $\quad b_{\leq^*} = b \quad d_{\leq^*} = d$



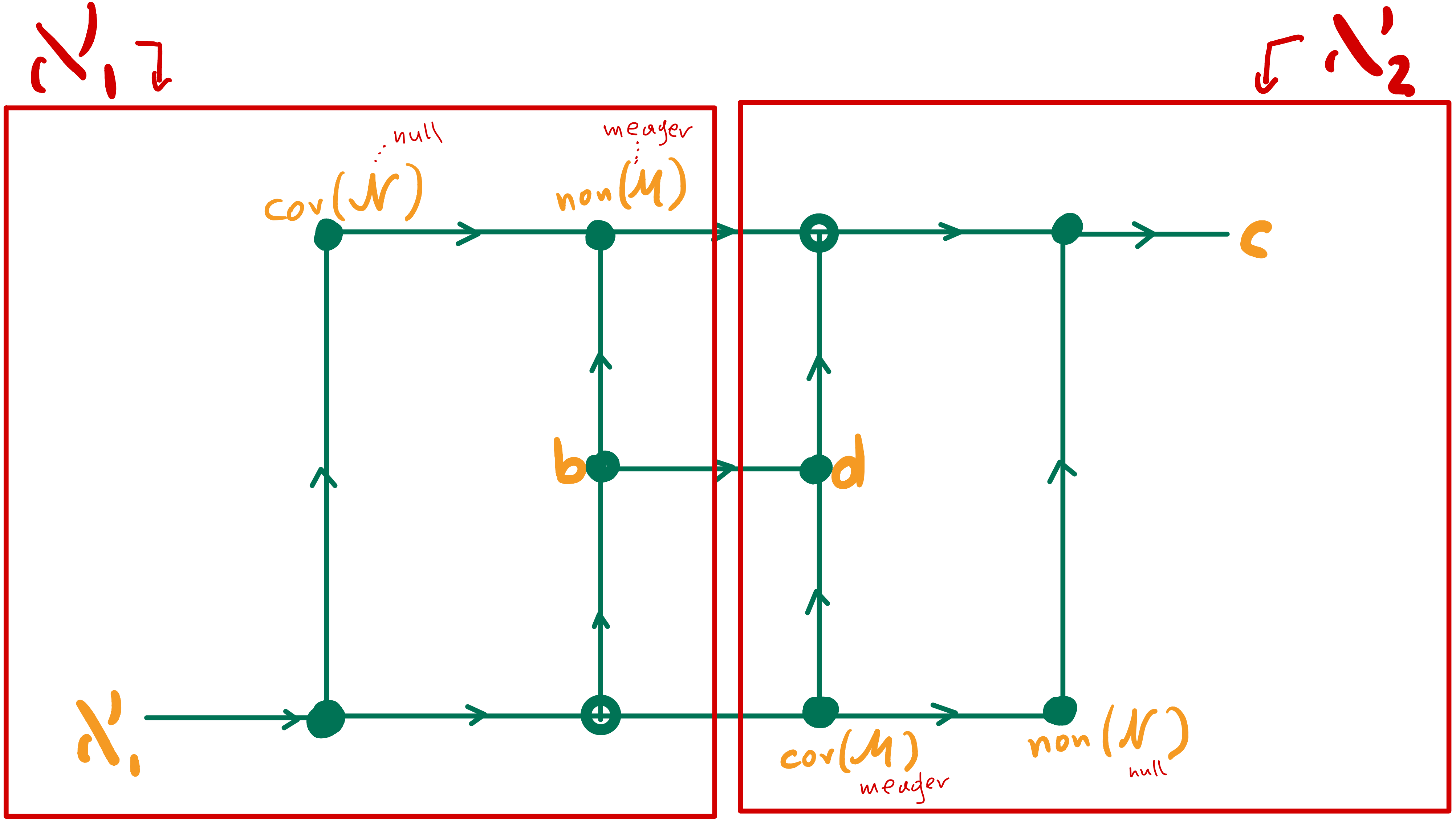
$f R_{\mathrm{non}\mathcal{M}}\, g : \forall^\infty n : f(n) \neq g(n)$ $\quad b_{R_{\mathrm{non}}} = \mathrm{non}(\mathcal{M}) \quad d_{R_{\mathrm{non}}} = \mathrm{cov}(\mathcal{M})$

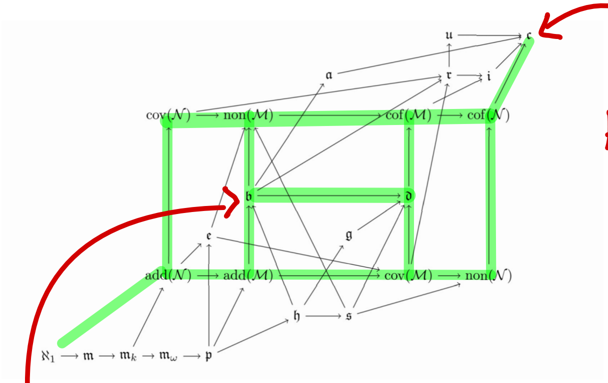
Cichoń's Diagram, assuming CH



Cichoń's Diagram, assuming MA + ¬CH

Cohen's universe Con(ZFC + $2^{\aleph_0} = \aleph_2$))





$c = 2^{\aleph_0}$

How many reals?

$(2^\omega, \mathbb{R}, \omega^\omega, \dots)$

$\mathfrak{b} =$ How many functions $f: \omega \to \omega$ do you need so that no $g: \omega \to \omega$ dominates all of them?

$f \leq^* g \iff \exists n_0 \forall n \geq n_0 \; f(n) \leq g(n)$

How to increase b.

$\overset{\text{consistent}}{\text{Con}}(\mathfrak{b} \geq \aleph_2)$:

Iterate forcing extensions:

$$\underbrace{V, V^{Q_0}, V^{Q_0 * Q_1}, \dots\dots}_{\omega_2 \text{ many}}$$

In each step add a real that dominates the previous universe.

In the end, every (really?) set of size $\aleph_1$ is dominated.

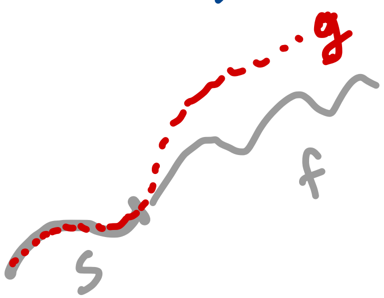
Hechler forcing

Successor step for increasing $\mathfrak{b}$:

$$\mathbb{P} = \{ (s,f) : s \in \omega^{<\omega},\ f \in \omega^\omega,\ \forall k\ s(k) \geq f(k) \}$$

$$(s',f') \lesssim (s,f) \iff s' \supseteq s,\ f' \geq f$$

everywhere



So $\forall k \in \mathrm{dom}(s')$: $s'(k) \geq f'(k) \geq f(k)$

$$(s,f) \Vdash g \supseteq s,\ g \geq f$$

Hence g dominates all old functions.

Limit step ω:

When we have defined

$$\begin{tikzcd} \mathbb{P}_0 \arrow[r, phantom, "\subseteq"] \arrow[d, dash, dotted] & \mathbb{P}_1 \arrow[r, phantom, "\subseteq"] \arrow[d, dash, dotted] & \mathbb{P}_2 \arrow[r, phantom, "\subseteq"] \arrow[d, dash, dotted] & \dots \\ \{1\} & \mathbb{P}_0 * \underset{\sim}{\mathbb{Q}}_0 & \mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_1 & \end{tikzcd}$$

Let $\mathbb{P}_\omega = \bigcup_{n<\omega} \mathbb{P}_n$ „direct limit" „finite support limit"

not really!

"All" nice properties of the $\mathbb{P}_n$ are inherited by $\mathbb{P}_\omega$.

Result: Con$(\mathfrak{b} = \mathfrak{c} = \aleph_2)$

Similarly Con$(\mathfrak{b} = \mathfrak{c} = \kappa)$ for regular κ.

(What happens in limit stages?)

$$\begin{tikzcd}
\mathbb{P}_0 \arrow[r, phantom, "\subseteq"] \arrow[d, dotted, no head] & \mathbb{P}_1 \arrow[r, phantom, "\subseteq"] \arrow[d, dotted, no head] & \mathbb{P}_2 \arrow[r, phantom, "\subseteq"] \arrow[d, dotted, no head] & \cdots \\
\{1\} & \mathbb{P}_0 * \underset{\sim}{\mathbb{Q}}_0 & \mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_1 &
\end{tikzcd}$$

⊕

If all $\underset{\sim}{\mathbb{Q}}_i$ are ccc, then all $\mathbb{P}_i$ will be ccc.

If all $\underset{\sim}{\mathbb{Q}}_i$ are of size $\leq \aleph_1$, then all $\mathbb{P}_i$ $(i < \omega_2)$ are "morally" $\leq \aleph_1$

(dense subset)

(CH preserved)

$[\forall i: |\mathbb{Q}_i| = c \Rightarrow \mathbb{P}_\delta \Vdash c \leq |\delta| + \aleph_1]$

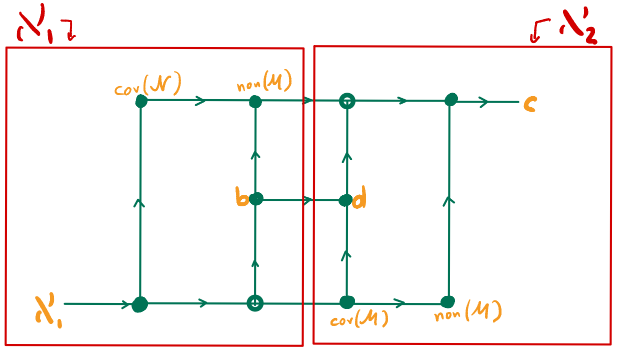
⊖

If all $\mathbb{Q}_i$ are nontrivial,

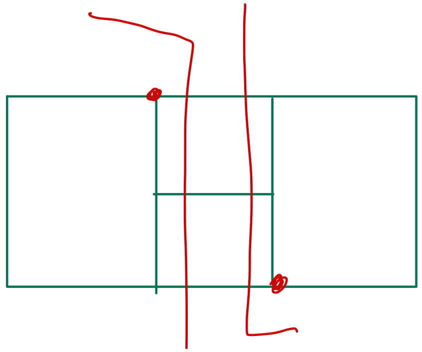
then $\mathbb{P}_{i+\omega}$ adds a Cohen real over $V^{\mathbb{P}_i}$

Consequences of Cohen reals

Cohen's universe



Finite support iteration, length δ_{limit}



$\mathrm{non}(\mathcal{M}) \leq \mathrm{cf}(\delta) \leq \mathrm{cov}(\mathcal{M})$

$\omega < \delta$ regular

$(c_i : i < \delta)$

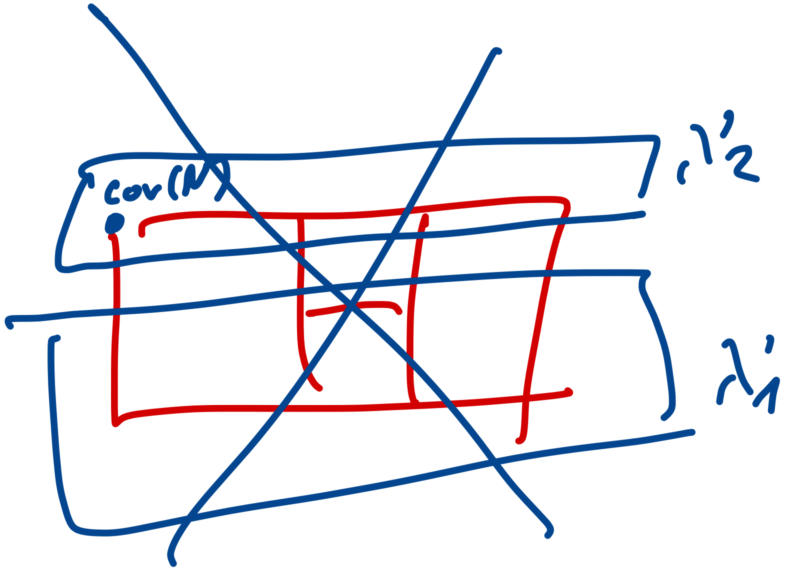
↑ ith limit

$\{c_i : i < \delta\}$ size δ

non meager!

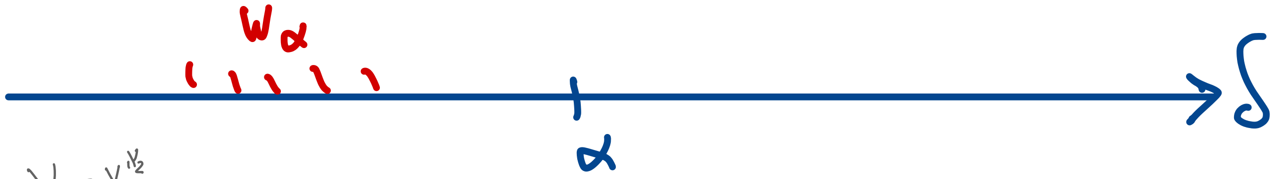
$\mathrm{non}(\mathcal{M}) \leq \delta$

$\mathrm{cov}(\mathcal{M}) \geq \delta$



How to get $\aleph_1 < b < c$? "SMALL FORCING"

E.g. $b = \aleph_3$, $c = \aleph_8$



Assume $V \models c \leq \aleph_8 = \aleph_8^{\aleph_2}$

Iterate forcing extensions $\delta := \omega_8$ many steps.

(In V:) For each $\alpha < \omega_8$, fix a set $W_\alpha \subseteq \alpha$ of size $< \aleph_3$ such that every element of $[\delta]^{<\aleph_3}$ is covered by some W_α.

Let $\Vdash_{\mathbb{P}_\alpha} \underset{\sim}{Q}_\alpha :=$ (Hechler forcing) $\cap$ (the universe Borel generated by $\{ g_\beta : \beta \in W_\alpha \}$)

g_β …… generic real, stage β

Then in $V[G_\delta]$, all sets of size $< \aleph_3$ will be dominated, so $\Vdash_{\mathbb{P}_\delta} b \geq \aleph_3$.

We know how to ensure $\Vdash_{\mathbb{P}_\delta} \mathfrak{b} \geq \aleph_3$: In each successor stage, take care of some set of size $< \aleph_3$, use a small Hechler forcing to dominate it.

How to ensure $\Vdash_{\mathbb{P}_\delta} \mathfrak{b} \leq \aleph_3$?

A "witness" for $\mathfrak{b} \leq \aleph_3$:= An unbounded set of size $\aleph_3$.

A $\boxed{\text{strong witness}}$ for $\mathfrak{b} \leq \aleph_3$: A set $S \subseteq \omega^\omega$, $|S| \geq \aleph_3$,

and: $\forall g \in \omega^\omega : \left|\{ s \in S : s \leq^* g \}\right| < \aleph_3$

(Any subset of size $\aleph_3$ will be a witness.

Note: If $|S| = \aleph_7$, S a strong witness for $\mathfrak{b} \leq \aleph_3$, then S is also a witness for $\mathfrak{d} \geq \aleph_7$.

We will show soon how to $\left\{ \begin{array}{l} \text{(a) get} \\ \text{(b) preserve} \end{array} \right\}$ strong witnesses.

Strong witness, examples:

A witness for $\mathrm{non}(\mathcal{M}) = \aleph_1$: $L = \{x_i : i < \omega_1\}$, not meager

A strong witness for $\mathrm{non}(\mathcal{M}) = \aleph_1$: $L = \{x_i : i < \omega_1\}$,
every meager subset is countable } Luzin set
Лузин

How to get strong witnesses:

Iterate Cohen reals, finite support, length $\lambda > \omega$

Get a sequence $\bar{c} = (c_i : i < \lambda)$.

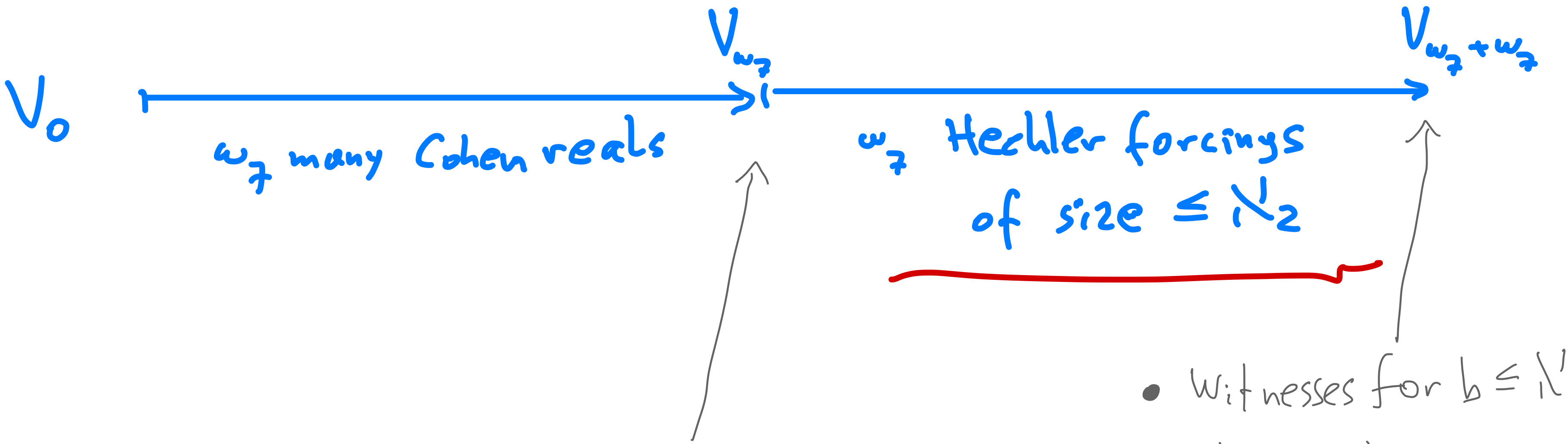
$\forall \kappa < \lambda$: κ regular $\Longrightarrow$ $\bar{c} \restriction \kappa$ is a strong witness

... for $\mathfrak{b} \leq \kappa$

(also for $\mathrm{non}(\mathcal{M}) \leq \kappa$, ...)

The plan to get $\aleph_1 < \underset{\aleph_3}{\underset{\shortparallel}{\mathfrak{b}}} < \underset{\aleph_7}{\underset{\shortparallel}{\mathfrak{c}}}$

(and many other $\mathfrak{b}_R$ in between):



Have strong witnesses for $\mathfrak{b} \leq \aleph_1$, $\mathfrak{b} \leq \aleph_2$, ... $\mathfrak{b} \leq \aleph_7$

also $\mathfrak{d} \geq \aleph_1$, ... $\mathfrak{d} \geq \aleph_7$

- Witnesses for $\mathfrak{b} \leq \aleph_2$ have disappeared
- $\Vdash \mathfrak{b} \geq \aleph_3$
- Have the strong witnesses for $\mathfrak{b} \leq \aleph_3$ survived? $\mathfrak{d} \geq \aleph_7$

DEFINITION: (λ regular $> \aleph_0$. For simplicity: successor)

$\mathbb{P}$ is **λ-good** (for $\leq^*$) if: (λ'-good for all $\lambda' \geq \lambda$)

For all $\mathbb{P}$-names $\underset{\sim}{z}$

there is a set $Y \in V$, $|Y| < \lambda$ of "representatives"

such that:

$\forall x$: If Y does not bound x, then neither does $\underset{\sim}{z}$.

$\exists y\ x \leq y \leftarrow x \leq \underset{\sim}{z}$

$$\forall x: \left[\left(\forall y \in Y: x \not\leq^* y \right) \Rightarrow \Vdash_{\mathbb{P}} x \not\leq^* \underset{\sim}{z} \right]$$

Example: If $|\mathbb{P}| < \lambda$.

$(p_k)_{k \in \omega}$ $\quad p_k \Vdash \underset{\sim}{z} \restriction k = z^* \restriction k$

Proof: Let $N \prec H(\chi)$, $|N| = |\mathbb{P}|$, $\mathbb{P} \cup \{\mathbb{P}\} \subseteq N$. $\underset{\sim}{z} \in N$

Let $Y := \omega^\omega \cap N$.

Assume x: $p \Vdash x \leq^* \underset{\sim}{z}$. Wlog $p \Vdash x \leq_0 \underset{\sim}{z}$. (Note: $x \notin N$)

In N, interpret $\underset{\sim}{z}$ as z^*. Then $x \leq z^* \in Y$, ↯.

$p_k \Vdash x \restriction k \leq \underset{\sim}{z} \restriction k = z^* \restriction k$

DEFINITION: (λ regular $> \aleph_0$. For simplicity: successor)

$\mathbb{P}$ is **λ-good** (for $\leq^*$) if:

For all $\mathbb{P}$-names $\underset{\sim}{z}$ there is a set $Y \in V$, $|Y| < \lambda$ of "representatives" such that:

$\forall x$: If Y does not bound x, then neither does $\underset{\sim}{z}$.

$$\forall x: \left[(\forall y \in Y: x \not\leq^* y) \Rightarrow \Vdash_{\mathbb{P}} x \not\leq^* \underset{\sim}{z} \right]$$

Fact: (1) If $\bar{c} = (c_i : i < \kappa)$ is a strong witness for $\mathfrak{b} \leq \kappa$

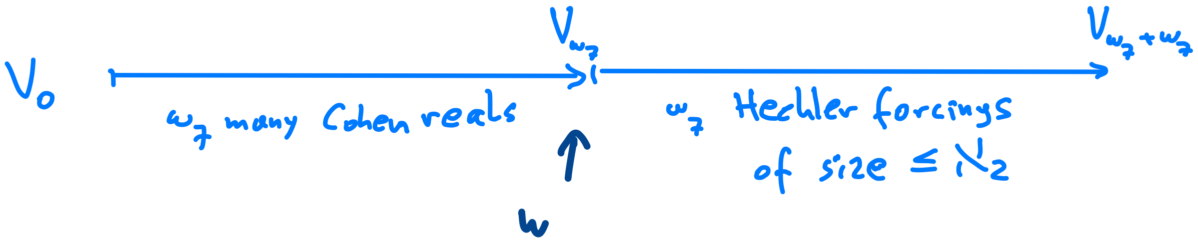
[every $\leq^*$-bounded subset is $< \kappa$]

and if $\mathbb{P}$ is λ-good for some $\lambda \leq \kappa$

(so also κ-good)

then $\Vdash_{\mathbb{P}} (c_i : i < \kappa)$ is still a strong witness

(2) Goodness is preserved in FS iteration.



Reformulation of λ^+-goodness:

For all $\mathbb{P}$-names $\underset{\sim}{z}$
there is a set $Y \in V$, $|Y| < \lambda$ of "representatives"

such that:

$\forall x$: If Y does not bound x, then neither does $\underset{\sim}{z}$.

$\exists y\ x \leq y \longleftarrow x \leq \underset{\sim}{z}$

$$\forall x : \left[\left(\forall y \in Y : x \not\leq^* y \right) \Rightarrow \Vdash_{\mathbb{P}} x \not\leq^* \underset{\sim}{z} \right]$$

For all $N \prec H(\chi)$ of size λ, with $\mathbb{P} \in N$:

for all $\underset{\sim}{z} \in N$ (so $\underset{\sim}{z}[G] \in N[G]$):

$\forall x$: If $\Vdash x \leq^* \underset{\sim}{z}$, then $x \leq^* \omega^\omega \cap N =: Y$

i.e. $\exists y \in Y : x \leq^* y$

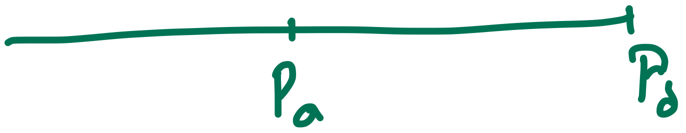
Def Let $p \Vdash \underset{\sim}{x} \in \omega^\omega$. An interpretation of $\underset{\sim}{x}$ (below p) is a pair $(x^*, \bar{p})$,

$x^* \in \omega^\omega$, $\bar{p} = (p_n) \in \mathbb{P}^\omega$, decreasing,

$$\forall n \quad p_n \Vdash \underset{\sim}{x} \restriction n = \check{x}^* \restriction n$$

Quotients: FS iteration $(P_\beta, Q_\beta : \beta < \delta)$, limit P_δ.

$\alpha < \delta$.



$\{p \restriction [\alpha, \delta] : p \restriction \alpha \in G\}$

$$\Vdash_{P_\alpha} P_\delta : P_\alpha := \{p \in P_\delta : p \restriction \alpha \in G_{P_\alpha}\}$$

Then $\Vdash P_\delta : P_\alpha$ is equivalent to a FS iteration indexed by $\delta \setminus \alpha$ of forcings $\underline{Q_\beta}$

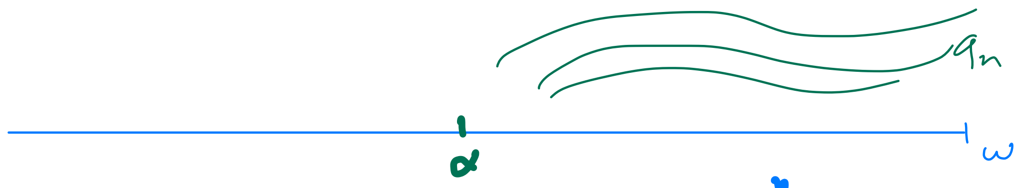
Preservation of goodness in limit stages δ

(successors: homework)

Main case: $cf(\delta) = \omega$.

$x \le_n y \iff$
$\forall k \ge n \ x(k) \le y(k)$
$x \le^* y \iff \exists n \ x \le_n y$

$\underset{\sim}{z} \in N$



$\underset{\sim}{z}[G] \in N[G]$

x arbitrary
$\notin N$?

Assume $\Vdash_{\mathbb{P}_\delta} x \le^* \underset{\sim}{z}$, but $x \not\le^* Y := \omega^\omega \cap N$

Find p, n: $p \Vdash x \le_n \underset{\sim}{z}$, $p \in \mathbb{P}_\alpha$, $\alpha < \delta$.
$\notin N$

Work in $V[G_\alpha]$. Find (q_n) new interpreting $\underset{\sim}{z}$ as $\underset{\sim}{z}^*$. $\leftarrow \mathbb{P}_\delta : \mathbb{P}_\alpha$

($\underset{\sim}{z}^*$ is a $\mathbb{P}_\alpha$-name)
in N!!

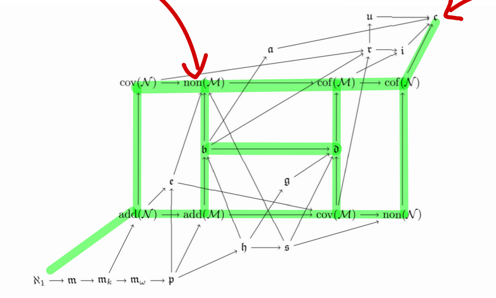
In $V[G_\alpha]$, $x \not\le^* z^*$ (as $\mathbb{P}_\alpha$ is good)

Find $n' > n$ $x(n') > z^*(n')$.

$(p, q_{n'+1}) \Vdash x(n') > z^*(n') = \underset{\sim}{z}(n')$ ↯

(In general:
$\{z : x \not\le_n z\}$
is open)

$\mathrm{non}(\mathcal{M})$ = How many reals do you need so that no meager set contains all of them



$c = 2^{\aleph_0}$

How many reals ?

$(2^\omega, \mathbb{R}, \omega^\omega, \ldots)$

$M_g = \{ f \in \omega^\omega \mid \forall^\infty n \; f(n) \neq g(n) \}$ meager

How to increase ~~$\mathfrak{b}$~~. $\mathrm{non}(\mathcal{M})$

$$\mathrm{Con}\left(\overset{\mathrm{non}(\mathcal{M})}{\cancel{\mathfrak{b}}} \geq \aleph_2 \right):$$

Iterate forcing extensions:

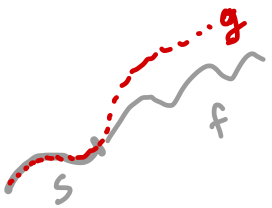
$$\underbrace{V, V^{Q_0}, V^{Q_0 * Q_1}, \ldots\ldots}_{\omega_2 \text{ many}}$$

In each step add a real that ~~dominates~~ covers the previous universe. (with a meager set M_g)

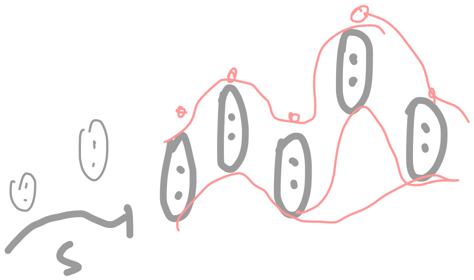
In the end, every (really?) set of size $\aleph_1$ is ~~dominated~~ covered, so $\mathrm{non}(\mathcal{M}) \geq \aleph_2$

How to cover the old reals with a meager set $M_g = \{f \in \omega^\omega : \forall^\infty_n \ f(n) \neq g(n)\}$

Hechler:



Eventually different:



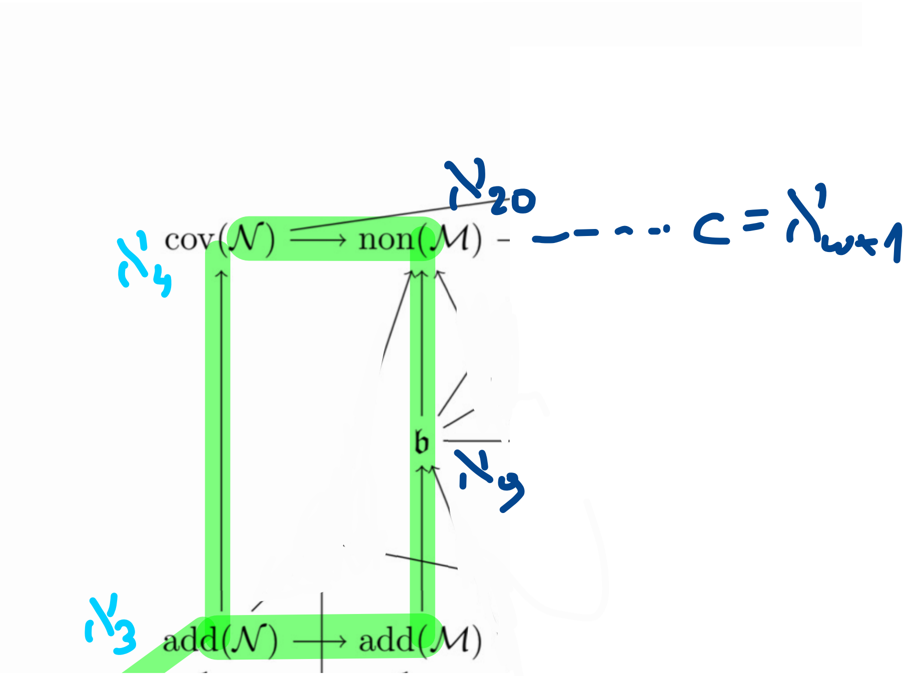
$$\mathbb{E} = \{ (s,k,\varphi) : s \in \omega^{<\omega},\ k \in \omega,\ \varphi : \omega \longrightarrow [\omega]^{\leq k},\ \forall i \in \mathrm{dom}(s) : s(i) \notin \varphi(i) \}$$

$$(s',k',\varphi') \leq (s,k,\varphi) \iff \begin{array}{l} s' \supseteq s \\ k' \geq k \\ \forall i\ \varphi'(i) \supseteq \varphi(i) \end{array}$$

The generic function g eventually avoids all $f \in V$.

$$(s,k,\varphi) \Vdash g \supseteq s,\ \forall i\ g(i) \notin \varphi(i) \}$$

The left side of Cichoń's diagram



[If additionally you want add$(\mathcal{N}) = \aleph_3$, cov$(\mathcal{N}) = \aleph_4$,
use Amoeba forcings of size $\aleph_2$
and random forcings of size $\aleph_3$.]

Plan:

To get

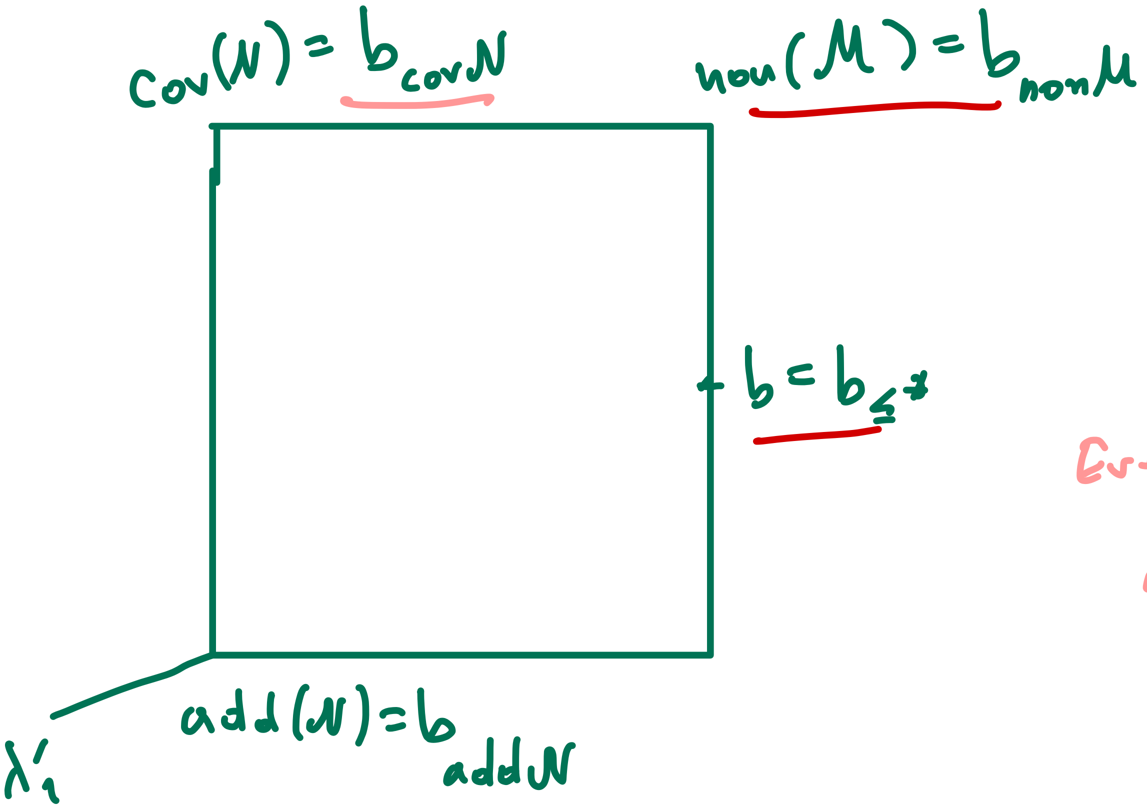
$$\begin{tikzcd}
\aleph_1 & < & b & < & \text{non}(\mathcal{M}) & < & c \\
& & \aleph_9 \arrow[u, dotted] & & \aleph_{20} \arrow[u, dotted] & & \aleph_{\omega+3} \arrow[u, dotted]
\end{tikzcd}$$

use a finite support iteration of

(a) Hechler forcings of size $\aleph_8$

(b) eventually different forcings of size $\aleph_{19}$

(length: $\aleph_{\omega+3}$)



Every σ-centered forcing is good for $R_{\mathrm{cov}\mathcal{N}}$, $R_{\mathrm{add}\mathcal{N}}$

Every Boolean alg. admitting a finitely add measure is good for $R_{\mathrm{add}\mathcal{N}}$

$\mathbb{P}$ σ-centered: $\mathbb{P} = \bigcup_{k\in\omega} \mathbb{P}_k$

$\mathbb{P}_k$ centered: $\forall F \subseteq \mathbb{P}_k$ finite $\Rightarrow \exists q\ \forall f \in F\ q \leq f$

$$I_n = [2^n, 2^{n+1})$$

$$R = \{ (x,y) \in 2^\omega \times 2^\omega : \forall^\infty n \;\; x\restriction I_n \neq y\restriction I_n \}$$

$$N_g = 2^\omega \setminus \{ f \mid \forall^\infty n \;\; \underbrace{f\restriction I_n \neq g\restriction I_n} \}$$ ~~positive~~

measure

measure

The plan to get $\aleph_1 < \underset{\aleph_2}{\mathrm{add}(\mathcal{N})} < \underset{\aleph_5}{\mathrm{cov}(\mathcal{N})} < \underset{\aleph_9}{\mathfrak{b}} < \underset{\aleph_{12}}{\mathrm{non}(\mathcal{M})} < \underset{\aleph_{16}}{\mathfrak{c}}$

$$\begin{tikzcd} V_0 \arrow[r, "\omega_{12} \text{ many Cohen reals}"'] & V_{\omega_7} \arrow[r, "\text{alternate:}"'] & {} \end{tikzcd}$$

alternate:
- Amoeba forcings, size $\aleph_1$
- random forcings, size $\aleph_4$
- Hechler forcings, size $\aleph_8$
- Eventually diff, size $\aleph_{11}$

For each of the 4 relations R

- easily $b_R \geq$ target value
- goodness $\Longrightarrow$ $b_R \leq$ target value

plus: strong witnesses

Ultralimits

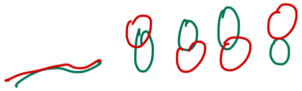
$\mathbb{E} = \{ (s,k,\varphi) : s \in \omega^{<\omega}, k \in \omega \quad \varphi : \omega \longrightarrow [\omega]^{\le k}, \forall i \in dom(s) : s(i) \notin \varphi(i) \}$

$(s',k',\varphi') \le (s,k,\varphi) \iff$ $s' \supseteq s$
$k' \ge k$
$\forall i \ \varphi'(i) \supseteq \varphi(i)$

The generic function g eventually avoids all $f \in V$.
$(s,k,\varphi) \Vdash g \supseteq s, \ \forall i \ g(i) \notin \varphi(i) \}$

Let D be an ultrafilter on ω.

$\bar{p} = (p_n : n \in \omega)$ a sequence
$p_n = (s, k, \varphi_n)$.



$q := \lim_D \bar{p} = (s, k, \varphi) \in \mathbb{E}$ $\quad j \in \varphi(i) \iff \{ n : j \in \varphi_n(i) \} \in D$

Fact $\forall A \in D : q \Vdash A \cap \{ n \in \omega : p_n \in G \}$ is infinite.

Thm If $\bar{f} = (f_i : i < \lambda)$ is a strong witness for $b \le \lambda$,
then $\mathbb{E}$ does not add a real dominating $\bar{f}$.

Thm If $\bar{f}=(f_i : i<\lambda)$ is a strong witness for $b \le \lambda$, then $\mathbb{E}$ does not add a real dominating $\bar{f}$.

Proof Assume $p \Vdash \forall i\ f_i \le^* \underset{\sim}{g}$.

$\forall i<\lambda\ \exists p_i, n_i \quad p_i \Vdash f_i \le_{n_i} \underset{\sim}{g}$. all $p_i = (s, \ell, \varphi_i)$

wlog all $n_i = n^*$ $\quad p_i \Vdash f_i \le_{n^*} \underset{\sim}{g}$.

Find $\underline{k} > n^*$ s.t. $\underline{(f_i(k) : i < \lambda)}$ is unbd. $\subseteq \omega$

Wlog $\quad f_i(k) \ge i$ for all $i < \underline{\omega}$.

Let $q = \lim_D (p_i)_{i<\omega}$.

$q \Vdash$ if $\underset{\substack{\uparrow \\ \infty \text{ many } i}}{p_i \in G}$, then $g(k) \ge \underset{\substack{\uparrow \\ g(k)=\infty}}{f_i(k)} \ge i$ $\quad \lightning$.

$R := \underline{\{(f,g) : \forall^\infty n \ f \restriction I_n \neq g \restriction I_n\}} \subseteq 2^\omega \times 2^\omega$

$I_n = [2^n, 2^{n+1})$

$N_f = \{g : f \not R g\}$ null set.

(If $(f_i : i < \lambda)$ unbd, then $\bigcup_i N_{f_i}$ cover

Assume $\underset{\sim}{y} \in N$, $\underline{x \text{ unbd by } N \cap 2^\omega}$

$P = \bigcup_k \mathbb{P}_k$

centered

$p_0 \Vdash x R_{n_0} \underset{\sim}{y}$ $\quad \forall n \geq n_0 \quad x \restriction I_n \neq y \restriction I_n$.

$\in \mathbb{P}_k$

$N \models$ For each $n \geq n_0$ find $s_n \in 2^{I_n}$ s.t. $\forall q \in \mathbb{P}_k : q \not\Vdash y \restriction I_n \neq s_n$

$s := \bigcup s_n \in N$!!

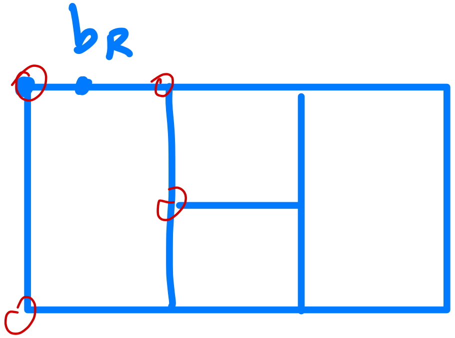
$\underline{x \text{ unbd by } s}$. Find large n

$x \restriction I_n = s \restriction I_n = s_n$

Can force $\underset{\sim}{y} \restriction I_n = s_n$. ↯

$\operatorname{cov}(\mathcal{N}) \leq b_R$

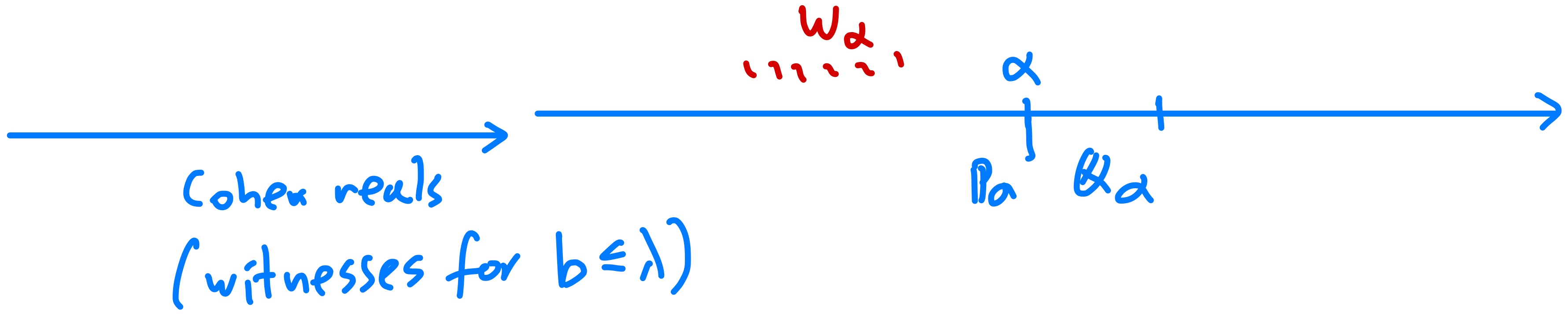
Correction $_{\mathrm{cov} N}$



$$I_n = [2^n, 2^{n+1})$$

$$f : \omega \to 2$$

$$\{ g : \forall^\infty n \; \underbrace{f \restriction I_n \neq g \restriction I_n}_{1 - \frac{1}{2^n}} \}$$ ~~positive~~ measure 1



For each α, $W_\alpha \subseteq \alpha$ is a "small" subset of α.

In $V[G_{\mathbb{P}_\alpha}]$, consider the (small!) universe V'_α generated by the generics at β, $\beta \in W_\alpha$.

This universe is created by a complete subforcing $\mathbb{P}_{W_\alpha} \lessdot \mathbb{P}_\alpha$. (i.e. is equal to $V[G_{\mathbb{P}_\alpha} \cap \mathbb{P}_{W_\alpha}]$)

In $V[G_{\mathbb{P}_\alpha}]$ we use the forcing (e.g.) $\mathbb{Q}_\alpha := \text{Hechler} \cap V'_\alpha$

|| The Hechler real added in stage α will dominate V'_α ||

$(c_i : i < \lambda_c)$

$(c_i : i < \lambda_b)$

$(c_i : i < \lambda_{non\, \mathcal{M}})$

R an F_σ relation, $\forall y$ $\{x : xRy\}$ meager.

linear cofinal unbounded

$\mathrm{LCU}_R(\mathbb{P}, \lambda)$:

$b_R \leq \lambda$

$d_R \geq \lambda$

There is a sequence $(\underset{\sim}{c}_\alpha : \alpha < \lambda)$ of $\mathbb{P}$-names which is forced to be a strong witness for $b_R \leq \lambda$, $d_R \geq \mathrm{cf}\,\lambda$

$(\Vdash_{\mathbb{P}} \forall y : \{\alpha : c_\alpha R y\}$ is bounded$)$

For singular λ: $\mathrm{LCU}(\mathbb{P}, \lambda) = \mathrm{LCU}(\mathbb{P}, \mathrm{cf}(\lambda))$

"almost nothing is bounded by y"

cone of bounds

$\mathrm{COB}_R(\mathbb{P}, \lambda, \mu)$:

$b_R \geq \lambda$

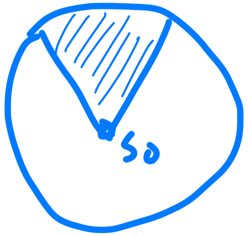
$d_R \leq \mu$

There is a $<\lambda$-directed partial order $(S, \leq)$ of cardinality μ and a sequence $(\underset{\sim}{d}_s : s \in S)$ of $\mathbb{P}$ names s.t.:

Usually $([\mu]^{<\lambda}, \subseteq)$

$$\forall \underset{\sim}{x}\ \exists s_0 :$$

$$\Vdash_{\mathbb{P}} \forall s \geq s_0 \quad \underset{\sim}{x}\, R\, d_s$$



"almost everything bounds x"

Remark: $\mathrm{LCU}(\mathbb{P}, \lambda)$ for R $= \mathrm{COB}(\mathbb{P}, \lambda, \lambda)$ for ¬Я witnessed by $(S, \leq) = (\lambda, \leq)$

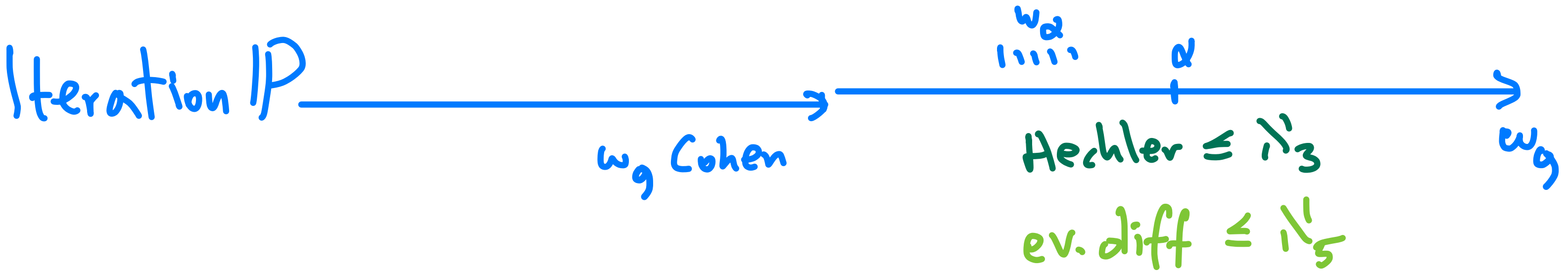
Example of LCU, COB:

$LCU_{\le^*}(\aleph_4)$

$COB_{\le^*}(\aleph_4, \aleph_9)$

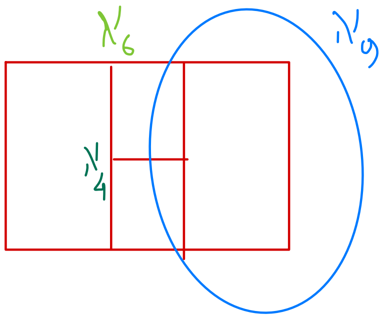
$LCU_{..}(\aleph_5)$

$COB_{..}(\aleph_6, \aleph_9)$



In stage $\alpha \ge \omega_9$, use a „small" set $W_\alpha \subseteq \alpha$ to define $\utilde{Q}_\alpha$

For the relation $\le^*$, define $\alpha \sqsubseteq \beta :\Longleftrightarrow Q_\alpha, Q_\beta = \text{Hechler}$, and $W_\alpha \subseteq W_\beta$.



This is $<\aleph_4$-directed

Let $d_\alpha :=$ Hechler real added by $\utilde{Q}_\alpha$.

Then $(\omega_9, \sqsubseteq, (d_\alpha)_\alpha)$ witnesses $COB(\mathbb{P}, \aleph_4, \aleph_9)$

(If $\utilde{x}$ is a name for a real, find α s.t. $\utilde{x}$ is in V_α'. For every β: $\alpha \sqsubseteq \beta \Rightarrow \utilde{x} \le^* \utilde{d}_\beta$)

For the relation R_{nonM}: similar

Fact: If κ strongly compact, θ regular $> \kappa$ (or just $\mathrm{cf}\,\theta > \kappa$)

then there is an elementary embedding $j: V \to M$

with the following properties: "j from κ to θ"

$\kappa = \mathrm{cp}(j)$

$M^{<\kappa} \subseteq M$ $\quad |y| < \kappa \Rightarrow j(y) = j''y$

$\mathrm{cf}\, j(\kappa) = \theta$ $\quad j(\kappa) \geq \theta$

$\forall \mu \;\; j(\mu) \leq \max(\mu, \theta)^{\kappa}$ $\quad$ If $\underline{\theta^{\kappa} = \theta}$, then $|j(\kappa)| = |j(\theta)| = \theta$

Whenever $(S, \leq)$ is $\leq\kappa$-directed,

then $j''S$ is cofinal in $j(S)$

Proof: Use $\mathrm{ro}\left(\mathrm{Fun}(\theta, \kappa, <\kappa)\right)$, Boolean ultrapower ...

Elements of $j(S)$ will be "averages" of $\leq \kappa$ elements of S.

Let $j: V \to M$ be an embedding from κ to θ

$P \models$ ccc.

- $j(P)$ is still ccc (is still a FS iteration)
- all $j(P)$-names of reals are in M (ctbly closed, ccc!)
- $M[G]^{<\kappa} \subseteq M[G]$ (in $V[G]$) [$G \subseteq j(P)$, V-gen and M-gen]
- $j''P$ is a complete subforcing of $j(P)$ ($\cong P$)

Let $j : V \to M$ be an embedding from κ to θ

$$LCU_R(P, \lambda) \Rightarrow LCU_R(j(P), j(\lambda))$$

$= cf(j(\lambda))$

(If $\lambda \neq \kappa$, λ regular, then $LCU(j(P), \lambda)$)

$V \models P \Vdash \bar{c} = (c_\alpha : \alpha < \lambda)$ strong witness

$M \models j(P) \Vdash j(\bar{c})$ (length $j(\lambda)$) is a strong witness.

"Every $j(P)$ name bounds only an initial segment of $j(\bar{c})$"

(ABSOLUTE between M and V)

$$COB(P, \lambda, \mu) \Rightarrow COB(j(P), \lambda, |j(\mu)|) \text{ if } \kappa > \lambda$$

witnessed by $j(P)$, $j(S)$ ..... λ-closed

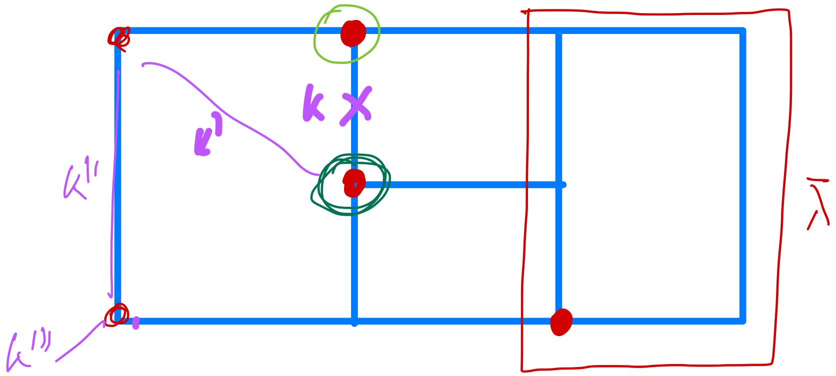
If $\kappa < \lambda$, S is $<\lambda$-closed

then $j''S$ is cofinal in $j(S)$

$$COB(j(P), \lambda, \mu) \text{ if } \kappa < \lambda$$

witnessed by $j''S$

Let $\mathbb{P}$ be a forcing, $\Vdash_{\mathbb{P}} \mathfrak{b} = \lambda_{\mathfrak{b}}$, $\Vdash \mathrm{non}\mathcal{M} = \lambda_{\mathrm{non}\mathcal{M}}$



$\Vdash$ right side $= \bar{\lambda}$

In V, let κ, κ' be strongly compact cardinals $\kappa' < \lambda_{\mathfrak{b}} < \kappa < \lambda_{\mathrm{non}\mathcal{M}}$

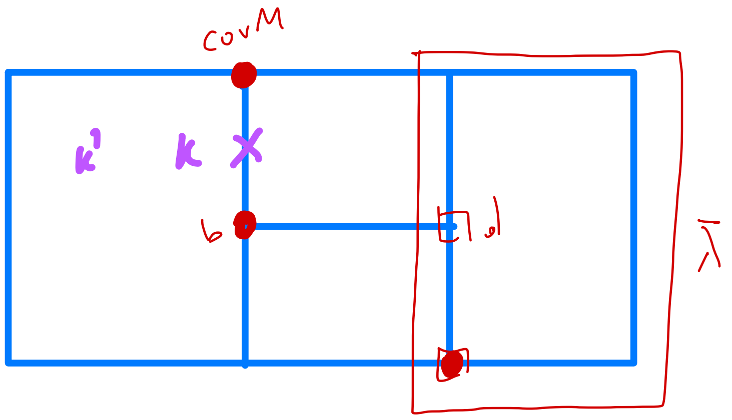
Assume $\mathrm{LCU}_{\mathrm{non}\mathcal{M}}(\mathbb{P}, \gamma)$ holds for all $\gamma \in [\lambda_{\mathrm{non}\mathcal{M}}, \bar{\lambda}]$ $\quad \mathrm{non}\mathcal{M} \leq \lambda_{\mathrm{non}\mathcal{M}}$, $\mathrm{cov}\mathcal{M} \geq \bar{\lambda}$

$\mathrm{LCU}_{\mathfrak{b}}(\mathbb{P}, \gamma)$ holds for all $\gamma \in [\lambda_{\mathfrak{b}}, \bar{\lambda}]$ $\quad \mathfrak{b} \leq \lambda_{\mathfrak{b}}$, $\mathfrak{d} \geq \bar{\lambda}$

$\mathrm{COB}_{\mathrm{non}\mathcal{M}}(\mathbb{P}, \lambda_{\mathrm{non}\mathcal{M}}, \bar{\lambda})$ holds $\quad \mathrm{non}\mathcal{M} \geq \lambda_{\mathrm{non}\mathcal{M}}$ $(\mathrm{cov}\mathcal{M} \leq \bar{\lambda})$

$\mathrm{COB}_{\mathfrak{b}}(\mathbb{P}, \lambda_{\mathfrak{b}}, \bar{\lambda})$ holds $\quad \mathfrak{b} \geq \lambda_{\mathfrak{b}}$ $(\mathfrak{d} \leq \bar{\lambda})$

We will consider ultrapowers $j_\kappa(\mathbb{P})$ and then $j_{\kappa'}(j_\kappa(\mathbb{P}))$ (in V!)



Assume

$LCU_{nonM}(\mathbb{P}, \gamma)$ holds for all $\gamma \in [\lambda_{nonM}, \bar{\lambda}]$ $nonM \leq \lambda_{nonM}$ $covM \geq \bar{\lambda}$

$LCU_{\leq^*}(\mathbb{P}, \gamma)$ holds for all $\gamma \in [\lambda_b, \bar{\lambda}]$ $b \leq \lambda_b$ $d \geq \bar{\lambda}$

$COB_{nonM}(\mathbb{P}, \lambda_{nonM}, \bar{\lambda})$ holds $nonM \geq \lambda_{nonM}$ $(covM \leq \bar{\lambda})$

$COB_b(\mathbb{P}, \lambda_b, \bar{\lambda})$ holds $b \geq \lambda_b$ $(d \leq \bar{\lambda})$

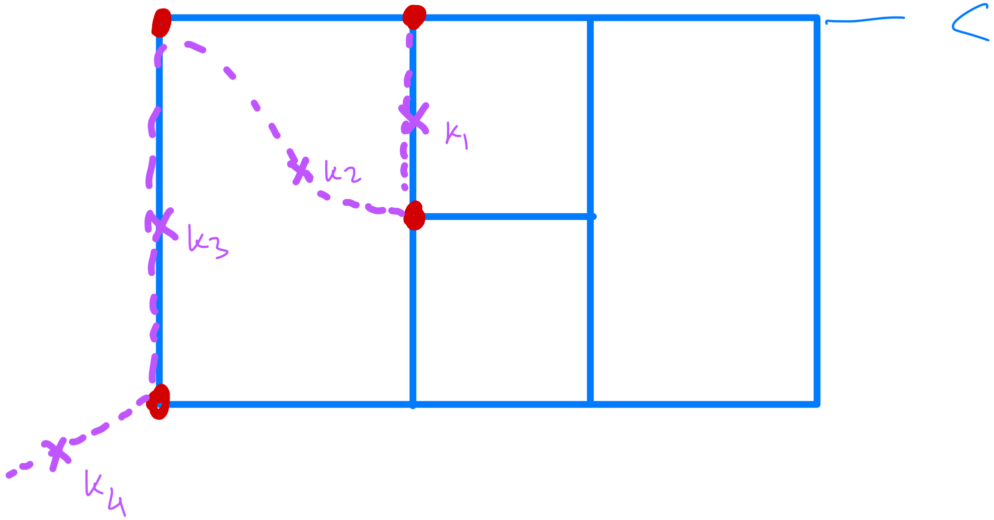
Let $j : V \to M_1$ go from κ to some λ_d.

Let $j' : V \to M_2$ go from κ' to some λ_{nonN}

$LCU(\mathbb{P}, \lambda_b)$	$LCU(\mathbb{P}, \kappa)$ $\textcircled{!}$	$COB(\mathbb{P}, \lambda_b, \bar{\lambda})$
$LCU(j(\mathbb{P}), \lambda_b)$	$LCU(j(\mathbb{P}), \lambda_d)$	$COB(\mathbb{P}, \lambda_b, \lambda_d)$
$LCU(j'(j(\mathbb{P})), \lambda_b)$	$LCU(j'(j(\mathbb{P})), j(\lambda_d))$	$COB(\mathbb{P}, \lambda_b, \lambda_d)$
$\underline{b \leq \lambda_b}$	$\underline{d \geq \lambda_d}$	$b \geq \lambda_b,\ d \leq \lambda_d$

$\mathrm{LCU}(\mathbb{P}, \lambda_b)$	$\mathrm{LCU}(\mathbb{P}, k)$	$\mathrm{COB}(\mathbb{P}, \lambda_b, \underset{(!)}{\bar{\lambda}})$
$\mathrm{LCU}(j(\mathbb{P}), \lambda_b)$	$\mathrm{LCU}(j(\mathbb{P}), \overset{(!)}{\lambda_d})$	$\mathrm{COB}(\mathbb{P}, \lambda_b, \lambda_d)$
$\mathrm{LCU}(j'(j(\mathbb{P})), \lambda_b)$	$\mathrm{LCU}(j'(j(\mathbb{P})), \lambda_d)$	$\mathrm{COB}(\mathbb{P}, \lambda_b, \lambda_d)$
$b \leq \lambda_b$	$d \geq \lambda_d$	$b \leq \lambda_b, \ d \geq \lambda_d$

$\mathrm{LCU}(\mathbb{P}, \lambda_{nonM})$	$\mathrm{LCU}(\mathbb{P}, \bar{\lambda})$	$\mathrm{COB}(\mathbb{P}, \lambda_{non}, \bar{\lambda})$
$\mathrm{LCU}(j(\mathbb{P}), \lambda_{nonM})$	$\mathrm{LCU}(j(\mathbb{P}), \bar{\lambda})$	$\mathrm{COB}(\mathbb{P}, \lambda_{non}, \bar{\lambda})$
$\mathrm{LCU}(j'(j(\mathbb{P})) \lambda_{nonM})$	$\mathrm{LCU}(j'(j(\mathbb{P})), \bar{\lambda})$	$\mathrm{COB}(\mathbb{P}, \lambda_{non}, \bar{\lambda})$
$nonM \leq \lambda_{nonM}$	$covM \geq \bar{\lambda}$	$nonM \geq \lambda_{non}, \ covM \leq \bar{\lambda}$



$j_4(j_3(j_2(j_1(\mathbb{P}))))$

$$\begin{tikzcd}
 & & & & \mathfrak{c}^{\mathrm{pre}}=\theta_\infty \arrow[d, equal] \\
 & \mathrm{cov}(\mathcal{N})^{\mathrm{pre}}=\theta_4 \arrow[r, dotted, no head] \arrow[dr, "\theta_3"'] & \mathrm{non}(\mathcal{M})^{\mathrm{pre}}=\theta_0 \arrow[r, dotted, no head] \arrow[urr] & \square \arrow[r, equal] \arrow[d, equal] & \square \arrow[dd, equal] \\
\mathbb{P}^{pre} \Vdash & & \mathfrak{b}^{\mathrm{pre}}=\theta_2 \arrow[u, "\theta_1"'] \arrow[r, dotted, no head] \arrow[d, equal] & \square \arrow[d, equal] & \\
 & \mathrm{add}(\mathcal{N})^{\mathrm{pre}}=\theta_6 \arrow[uu, "\theta_5"] \arrow[r, dotted, no head] & \square \arrow[r, dotted, no head] & \square \arrow[r, equal] & \square \\
\mathbb{P}^{fin} := \mathbb{P}^{pre} \cap N & & & & \mathfrak{c}^{\mathrm{fin}}=\lambda_\infty \arrow[ulll, "\theta_7"] \\
 & \mathrm{cov}(\mathcal{N})^{\mathrm{fin}}=\lambda_5 \arrow[r, dotted, no head] \arrow[dr] & \mathrm{non}(\mathcal{M})^{\mathrm{fin}}=\lambda_1 \arrow[r, dotted, no head] \arrow[ddr] & \square \arrow[r, dotted, no head] \arrow[d, equal] & \mathrm{cof}(\mathcal{N})^{\mathrm{fin}}=\lambda_6 \arrow[u] \\
\mathbb{P}^{fin} \Vdash & & \mathfrak{b}^{\mathrm{fin}}=\lambda_3 \arrow[u] \arrow[r, dotted, no head] \arrow[d, equal] & \mathfrak{d}^{\mathrm{fin}}=\lambda_2 \arrow[dr] & \\
 & \mathrm{add}(\mathcal{N})^{\mathrm{fin}}=\lambda_7 \arrow[uu] \arrow[r, dotted, no head] & \square \arrow[r, dotted, no head] & \mathrm{cov}(\mathcal{M})^{\mathrm{fin}}=\lambda_0 \arrow[u] \arrow[r, dotted, no head] & \mathrm{non}(\mathcal{N})^{\mathrm{fin}}=\lambda_4 \arrow[uu] \\
 & \aleph_1 \arrow[u] & & &
\end{tikzcd}$$

Def:

Let $S = (S, \le)$ be a p.o.

additivity, $b_\le$

directedness,

completeness,

$$\operatorname{add}(S) = \min\{\,|B| : B \subseteq S,\ B \overset{\text{no strict upper bound}}{\text{unbounded}}\,\}$$

$$\operatorname{cof}(S) = \min\{\,|D| : D \subseteq S,\ D \text{ dominates}\,\}$$

cofinality,

$d_\le$

If $S' \subseteq S$ is cofinal, then $\operatorname{add}(S) = \operatorname{add}(S')$

$\operatorname{cof}(S) = \operatorname{cof}(S')$

$S \approx S'$

usually $\lambda \ll \theta$

Definition: (1) A (θ,λ)-sequence is a sequence $(N_\alpha : \alpha < \lambda)$ of elementary submodels of "the universe" such that $\forall \alpha<\lambda$:

- $|N_\alpha| = \theta$, $N_\alpha^{<\theta} \subseteq N_\alpha$
- N_α contains "everything so far" (in particular $(N_\beta : \beta<\alpha)$ and $\theta \cup \{\theta\}$)

(2) N is a (θ,λ) model $:\Longleftrightarrow \exists (N_\alpha : \alpha<\lambda)$ as above, $N = \bigcup_{\alpha<\lambda} N_\alpha$

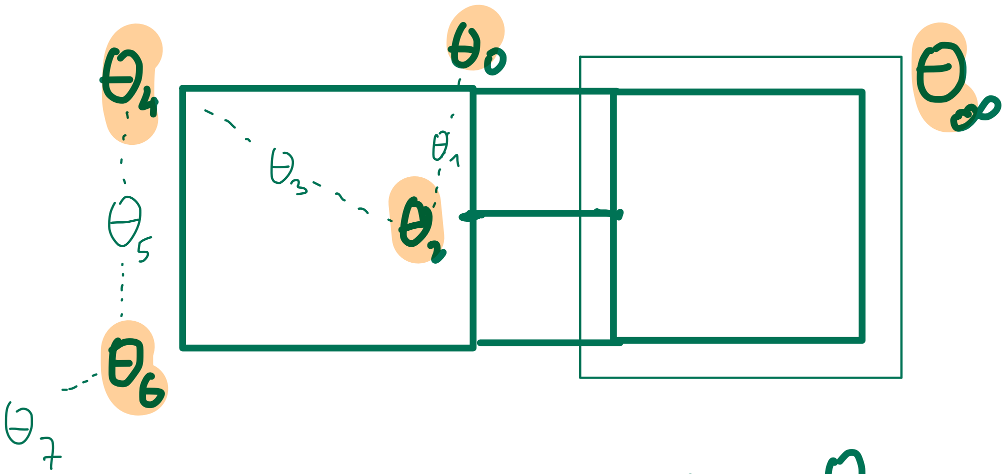
Fact: If N is a (θ,λ) model, $S \in N$:

⊙ $\mathrm{add}(S \cap N) \geq \min(\theta, \mathrm{add}(S))$

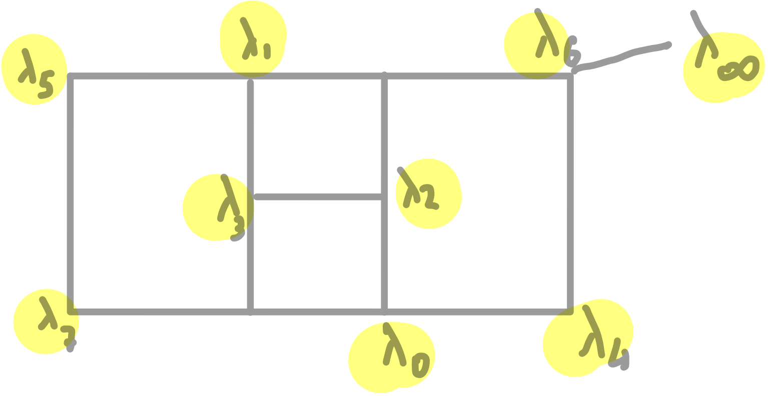
⊙ If $\mathrm{cof}(S) \leq \theta$, then $S \cap N \approx S$

⊙ If $\mathrm{add}(S) > \theta$, then $\mathrm{cof}(S \cap N) = \lambda$ $(= \mathrm{add}(S \cap N))$

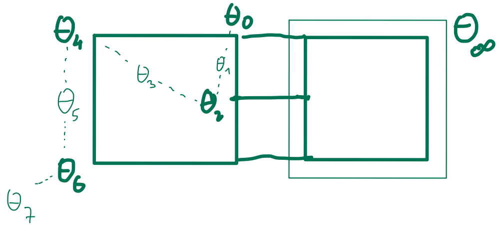
We start with a forcing notion $\mathbb{P}^{pre}$ which forces different values to the left side of Cichoń's Diagram, with strong witnesses (LCU, COB):



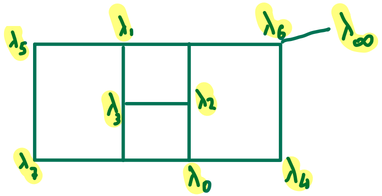
Our target values will be below θ_7:



Let N^0 be a (θ_0, λ_0)-model, limit of $(N^0_\alpha : \alpha < \lambda_0)$



Let N^1 be a (θ_1, λ_1)-model, limit of $(N^1_\alpha : \alpha < \lambda_1)$



$\vdots$ ↓ smaller models, but they contain all previous ones

Let N^7 be a (θ_7, λ_7)-model, limit of $(N^7_\alpha : \alpha < \lambda_7)$

$|N^8| = \lambda_\infty$, σ-closed

$$\text{Let } \mathbb{P}^{fin} = \underbrace{\underbrace{\mathbb{P}^{pre} \cap N^0}_{\mathbb{P}_0} \cap N^1}_{\mathbb{P}_1} \cap \dots \cap N^8$$

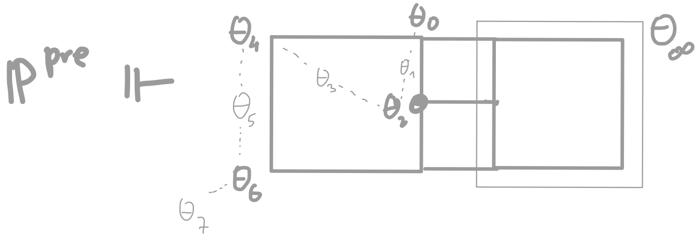
$\dots$

complete subforcings of $\mathbb{P}^{pre}$

Lemma

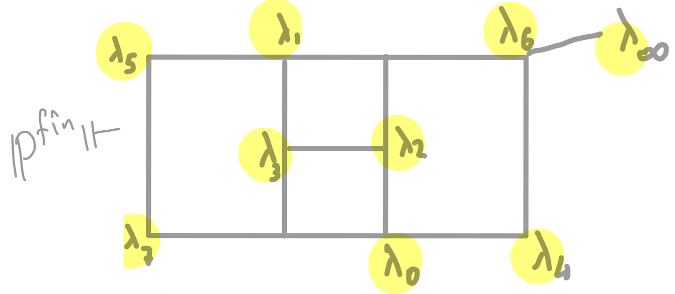
$$\underset{\lambda_0\text{- closed}}{\underset{\nearrow}{N^0}} \cap \underset{\lambda_1\text{- closed}}{\underset{\uparrow}{N^1}} = \bigcup_{(\alpha,\beta)\in\lambda_0\times\lambda_1} \underset{\theta_0\text{- closed}}{\underset{\uparrow}{N^0_\alpha}} \cap \underset{\theta_1\text{- closed}}{\underset{\uparrow}{N^1_\beta}}$$

$$N^0 \cap N^1 \cap N^2 = \bigcup_{\substack{(\alpha,\beta,\gamma)\in\\ \lambda_0\times\lambda_1\times\lambda_2}} N^0_\alpha \cap N^1_\beta \cap N^2_\gamma$$



Def: The LCU-spectrum Σ of a forcing notion (w.r.t. R) is the set of regular cardinals λ for which there is a sequence $(\underset{\sim}{c}_i : i<\lambda)$ of names s.t. $\Vdash_{\mathbb{P}}$ $\forall d: \{ i \mid \underset{\sim}{c}_i \, R \, d \}$ is bounded in λ.

(Recall: $\Vdash_{\mathbb{P}}$ $b_R \leq \min \Sigma$, $d_R \geq \sup \Sigma$)



Consider $R := \leq^*$

$$\Sigma(\mathbb{P}^{pre}) \supseteq \{ \overbrace{\theta_\infty, \theta_0, \theta_1, \theta_2}^{\text{models of size } \theta_0} \}$$

$$\mathbb{P}^0 = \mathbb{P}^{pre} \cap N_0 \Rightarrow \Sigma(\mathbb{P}^0) \supseteq \{ \overset{\lambda_0 \text{ many models}}{\lambda_0}, \theta_0, \theta_1, \theta_2 \}$$

$\vdots$

(θ_0, λ_0)-model

$\vdots$

$$\Sigma(\mathbb{P}^3) \supseteq \{ \lambda_0, \lambda_1, \lambda_2, \lambda_3 \}$$

$\vdots$

$$\Sigma(\mathbb{P}^{fin}) \supseteq \{ \lambda_0, \lambda_1, \overset{\max}{\lambda_2}, \overset{\min}{\lambda_3} \}$$

$b \leq \lambda_3$

$d \geq \lambda_2$

50% done!

Definition

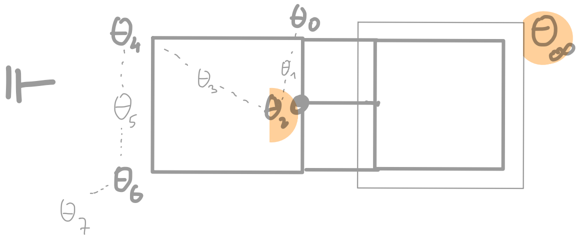
$\mathrm{COB}_R(\mathbb{P}, S) \iff S$ is a partial order,

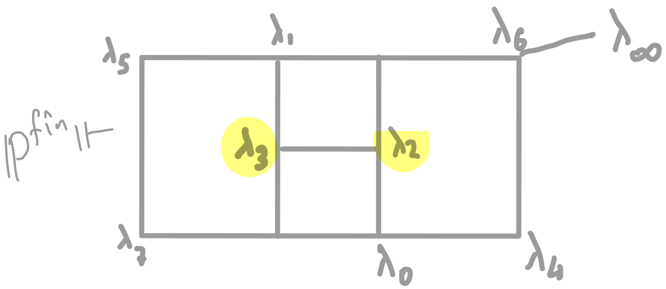
and there is a very dominating family $(\underset{\sim}{d}_s : s \in S)$

of names such that:

For all names $\underset{\sim}{x}$ there is $s_0 \in S$:

$$\Vdash_{\mathbb{P}} \forall s \geq s_0 : \underset{\sim}{x} \; R \; \underset{\sim}{d}_s$$





$\mathrm{COB}_{\leq^*}(\mathbb{P}, S)$ for a p.o. S with $\mathrm{add}(S) = \theta_2$

$\mathrm{cof}(S) = \theta_\infty$

$$\Downarrow$$

$\mathrm{COB}(\mathbb{P} \cap N, S \cap N)$.

What is $\mathrm{add}(S \cap N)$, $\mathrm{cof}(S \cap N)$?

$\geq \lambda_3$? $\leq \lambda_2$

Recall:

usually $\lambda \ll \theta$

Definition: (1) A (θ,λ)-sequence is a sequence $(N_\alpha : \alpha<\lambda)$ of elementary submodels of "the universe" such that $\forall\alpha<\lambda$:

- $|N_\alpha| = \theta$, $N_\alpha^{<\theta} \subseteq N_\alpha$
- N_α contains "everything so far" (in particular $(N_\beta : \beta<\alpha)$ and $\theta\cup\{\theta\}$

$(<\lambda$- closed$)$ $(<\theta$- closed$)$

(2) N is a (θ,λ) model $:\Longleftrightarrow \exists (N_\alpha : \alpha<\lambda)$ as above, $N = \bigcup_{\alpha<\lambda} N_\alpha$

Fact: If N is a (θ,λ) model, $S\in N$:

⊙ $\mathrm{add}(S\cap N) \geq \min(\theta, \mathrm{add}(S))$

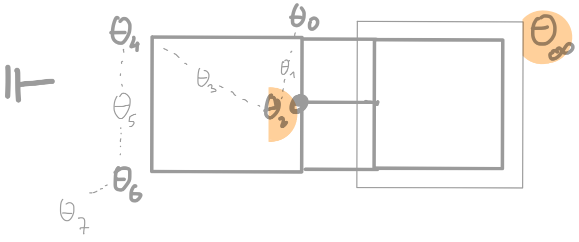
⊙ If $\mathrm{cof}(S) \leq \theta$, then $S\cap N \approx S$

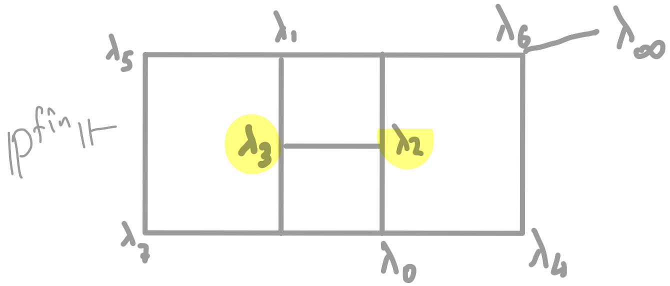
⊙ If $\mathrm{add}(S) > \theta$, then $\mathrm{cof}(S\cap N) = \lambda$ $(= \mathrm{add}(S\cap N))$

Consider $S_3 := S \cap N_0 \cap N_1 \cap N_2 \cap N_3$

Each N_i is $<\lambda_i$- closed

$$\underline{\mathrm{add}(S_3) \geq \min(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \lambda_3}$$





⊙ If $\mathrm{add}(S) > \theta$, then $\mathrm{cof}(S \cap N) = \lambda$ $(= \mathrm{add}(S \cap N))$

Let $\Lambda = \lambda_0 \times \lambda_1 \times \lambda_2$, size $= \lambda_2$

For $\eta = (\alpha, \beta, \gamma) \in \Lambda$,

let $N^\eta = N_{0,\alpha} \cap N_{1,\beta} \cap N_{2,\gamma}$ $\prec V$

This is θ_2- closed, so $\mathrm{add}(\underline{S \cap N^\eta}) \geq \theta_2 > \theta_3$

$\mathrm{cof}(\underline{S \cap N^\eta} \cap N_3) = \lambda_3$ witnessed by C_η.

$\bigcup_{\eta \in \Lambda} C_\eta$ has size $\lambda_2 \cdot \lambda_3 = \lambda_2$ and is cofinal

in $\bigcup_{\eta \in \Lambda} S \cap N^\eta \cap N_3 = S \cap (\bigcup_\eta N^\eta) \cap N_3 =$

$S \cap (N_0 \cap N_1 \cap N_2) \cap N_3$

For $\mathbb{P}^3 = \mathbb{P} \cap N_0 \cap N_1 \cap N_2 \cap N_3$

we get $\mathrm{COB}(\mathbb{P}^3, \overbrace{S \cap N_0 \cap N_1 \cap N_2 \cap N_3}^{\text{add} \geq \lambda_3,\ \text{cof} \leq \lambda_2})$,

so $\Vdash_{\mathbb{P}_3} \mathfrak{b} \geq \lambda_3, \ \mathfrak{d} \leq \lambda_2$.

Check that further intersections

$$S \cap N_0 \cap N_1 \cap N_2 \cap N_3 \cap \dots \cap N_8$$

keep a cofinal set, so add and cof do not change.

100% done!