Breaking and Preserving [Some] Choice

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Young Set Theory Workshop 2020

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Part I
Weak Choice Principles

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Let X be an infinite set, and let T be the set of all injective finite sequences of elements of X, ordered by end-extension.
Since X is infinite, T does not have any maximal nodes. If $C \subseteq T$ is an infinite chain, then $\bigcup C$ is an injective function from ω into X. □

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The following are equivalent:

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2. *Suppose that* A *is a set and* R *is a relation on* A *with* $\operatorname{dom} R = A$*, then given any* $a_0 \in A$*, there is a function* $f\colon \omega \to A$ *such that* $f(0) = a_0$ *and* $f(n)\ R\ f(n+1)$*.*

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3. *The Baire Category Theorem.*
4. *If* $\mathbb{P}$ *is a* σ*-closed forcing, then* $\mathbb{P}$ *is* σ*-distributive.*

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6. *Rasiowa–Sikorski Theorem.*
7. *A partial order* P *is well-founded if and only if it does not have infinite descending chains.*

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The **Axiom of Countable Choice** (AC_ω) states that if $\{A_n \mid n < \omega\}$ is a family of non-empty sets, then there is a function f with domain ω, and $f(n) \in A_n$ for all $n < \omega$.

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The inverse implication does not hold.

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Remark

The inverse implication does not hold. Namely, it is consistent with ZF *that* AC_ω *holds, but* DC *fails.*

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Let X be an infinite set, let $A_n = \{f : n+1 \to X \mid f \text{ is injective}\}$ for all $n < \omega$. Let f be a choice function from $\{A_n \mid n < \omega\}$ and let $f_n = f(n)$.

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Note that for any given n, f_i for $i < n$ can only enumerate less than $n! + 1$ elements, so by going to $f_{n!+1}$ we are guaranteed to find a suitable candidate for x_{n+1}. □

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Remark

The statement that every infinite set has a countably infinite subset is weaker than AC_ω*. It is equivalent to the statement "If* $\{A_i \mid i \in I\}$ *is a family of sets which are co-finite subsets of* $\bigcup A_i$*, then it admits a choice function."*

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6. *If* f *is a function between two metric spaces, then* f *is continuous if and only if it is sequentially continuous.*

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If F is a filter on a set X, consider the Boolean algebra $\mathcal{P}(X)/F$. By BPI it has a prime ideal, I. Define $A \in U \iff [X \setminus A]_F \in I$, then U is a filter extending F. Moreover, U is an ultrafilter, since for Boolean algebras prime ideals are maximal, and so either $[A]_F \in I$ or $[X \setminus A]_F \in I$. □

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Remark

The reverse implication holds. Namely, the Ultrafilter Lemma implies the Boolean Prime Ideal Theorem.

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Remark

BPI *is consistent with the existence of an infinite set without a countably infinite subset!*

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Remark

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8. *If* R *is a commutative ring with a unit, then every ideal is contained in a prime ideal.*

Part II
Symmetric Systems

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Lemma (The Symmetry Lemma)

$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x})$. □

Definition

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We say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a **symmetric system** when $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups on $\mathscr{G}$.

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We denote by $\mathsf{HS}_{\mathscr{F}}$ the class of all hereditarily $\mathscr{F}$-symmetric names.

Theorem

Let G be a V-generic filter, then $M = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}}\} = \mathsf{HS}_{\mathscr{F}}^G$ is a transitive class in $V[G]$ which contains V, and $M \models \mathsf{ZF}$.

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We will omit the subscripts from here on end, since the symmetric system will be clear from context.

Definition

We say that M satisfies **Small Violations of Choice** (SVC) if there is some X such that for any set Y there is some ordinal α and a surjection from $X \times \alpha$ onto Y.

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Let's see an example...

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If $\{\dot{x}_i \mid i \in I\}$ is a collection of names, we use $\{\dot{x}_i \mid i \in I\}^\bullet$ to denote the obvious name they define:

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This notation extends naturally to ordered pairs and functions, etc. For example, it simplifies $\check{x} = \{\check{y} \mid y \in x\}^\bullet$.

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$\pi p(\pi\alpha, n) = p(\alpha, n)$ by definition of the action of π. Therefore, $\pi p(\pi\alpha, n) = 1$ if and only if $p(\alpha, n) = 1$, and the equality follows. □

The filter $\mathscr{F}$ is defined to be generated by groups of the form $\operatorname{fix}(E) = \{\pi \in \mathscr{G} \mid \pi \upharpoonright E = \mathrm{id}\}$, where $E \subseteq \omega_1$ is a countable set.

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This is a fairly simplistic symmetric system.

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$$\mathbb{1} \Vdash^{\mathsf{HS}} \text{“}\dot{A} \text{ does not admit a choice function”}.$$

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$$\mathbb{1} \Vdash^{\mathsf{HS}} \dot{f}\colon \check{\omega} \to \bigcup_{i<\omega} \dot{A}_i \implies \exists n < \omega, \operatorname{rng} \dot{f} \subseteq \bigcup_{i<n} \dot{A}_i.$$

Part III
Preservation Theorems

Theorem

Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ *be a symmetric system. Suppose that* $\mathbb{P}$ *is* σ*-closed and* $\mathscr{F}$ *is* σ*-complete, then* $\mathbb{1} \Vdash^{\mathsf{HS}} \mathsf{DC}$.

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$$q \Vdash^{\mathsf{HS}} \{\dot{t}_n \mid n < \omega\}^\bullet \text{ is an infinite chain in } \dot{T}. \qquad \square$$

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Theorem (K.)

Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a mixable symmetric system admitting an absolute representative. Since $\breve{\omega}$ is injective and densely measurable, $\mathbb{1} \Vdash^{\mathsf{HS}} \mathsf{AC}_\omega$.

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Thank you
For
Your attention!

Some suggested reading...

1. K., **Iterating symmetric extensions**. *J. symb. log.* **84** (2019), 123–159 (arXiv:1606.06718).
2. K., **Preserving Dependent Choice**. *Bulletin Polish Acad. Sci. Math.* **67** (2019), 19–29 (arXiv:1810.11301).
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