Geometric Set Theory III

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This lecture is an introduction to Part II of the book, which is on balanced forcing extensions.

Most of the material is from Chapters 5 and 9.

A sentence is Σ^2_1 if it has the form $\exists A \subseteq \mathbb{R}\, \phi(A)$, where the quantifiers in ϕ range only over $\mathbb{R}$.

Many forms of the Axiom of Choice can be expressed as Σ^2_1 statements. For instance:

- The Continuum Hypothesis (CH)
- The existence of an ω_1-sequence of distinct reals
- The existence of a nonprincipal ultrafilter on ω
- The existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$
- The existence of a selectors for a given analytic equivalence relation
- The existence of ω-colorings of a given analytic graph on Euclidean space.

Part II of the book studies extensions of Choiceless models of ZF by forcings which add a Σ^2_1 consequence of Choice by countable approximations.

We analyze these models by considering amalgamation properties of these approximations.

Our approach can be seen as an elaboration of Shelah's 1985 paper "On Measure and Category."

Solovay models

Suppose that κ is a strongly inaccessible cardinal, and that

$$G \subseteq \mathrm{Col}(\omega, {<}\kappa)$$

is a V-generic filter. The resulting model

$$\mathrm{HOD}^{V[G]}_{V,\mathcal{P}(\omega)}$$

is a Solovay model (which we will call W).

We study forcing extensions of the Solovay model which recover forms of the Axiom of Choice.

The model $L(\mathbb{R})^{V[G]}$ is also called a Solovay model.

For all of what follows one can take W to be $L(\mathbb{R})^{V[G]}$.

The model $V[G]$ contains V-generic filters for each partial order in V_κ.

In particular, for partial orders of the form $\mathrm{Col}(\omega, {<}\gamma) * \dot{Q}$, for $\gamma < \kappa$, it contains filters of the form (K, H), where H is $V[K]$-generic for $\dot{Q}_K$.

Much of the analysis in part 2 of the book takes place in models of the form $V[K]$, where τ_K is the first coordinate of a balanced virtual condition.

In Solovay models:

- AC fails and $CC_{\mathbb{R}}$ holds;
- There are no ω_1-sequences of distinct reals;
- Every set of reals is Lebesgue measurable and has the property of Baire;
- Every ultrafilter on ω is principal (trivial).

$W[U]$

In the central example, we force over W with $\mathcal{P}(\omega)/\mathrm{Fin}$ (the order of infinite subsets of ω, under mod-finite containment) to add a nonprincipal ultrafilter U.

It turns out to be more natural to force with set of ω-sequences from $\mathcal{P}(\omega) \setminus \mathrm{Fin}$ whose ranges have the finite intersection property, with $p \leq q$ if the filter generated by the range of p contains the one generated by the range of q.

In the resulting model:

- (Henle-Mathias-Woodin) Every wellordered sequence in $W[U]$ consisting of elements of W is in W (so there are no new sets of ordinals)
- (Di Prisco - Todorcevic) There is no selector for $\mathbb{E}_0$ (mod-finite equivalence on $\mathcal{P}(\omega)$)

There are many open questions about this model, including:

- For which countable Borel equivalence relations E is there an injection from the E-classes to the $\mathbb{E}_0$-classes in $W[U]$?
- If X and Y are sets in W and X injects into Y in $W[U]$, must the same be true in W?
- Can one carry out the Banach-Tarski paradox in $W[U]$?
- For which equivalence relations can the classes be linearly ordered in $W[U]$? (Yes for $\mathbb{E}_0$, no for $\mathbb{F}_2$)
- What types of ultrafilters on ω exist in $W[U]$? Are they all rapid?

Suslin orders

A preorder $\langle P, \leq \rangle$ is *Suslin* if there is a Polish space X such that

- P is an analytic subset of X
- the ordering $\leq$ is an analytic subset of X^2
- (the incompatibility relation is an analytic subset of X^2)

We are mostly (but not only) interested in the case where P is σ-closed.

Examples of Suslin orderrs

- Countable partial functions from $\mathbb{R}$ to 2 (literally, functions with domain ω whose ranges are countable partial functions from $\mathbb{R}$ to 2), under the order of extension.
- Countable subsets of $\mathcal{P}(\omega)$ (disjoint with some Borel ideal I) with the finite intersection property, under the relation of generating a larger filter (e.g., $\mathcal{P}(\omega)/\mathrm{Fin}$).
- Countable partial selectors for a fixed analytic equivalence relation.
- Countable partial injections from the F-classes to the E-classes, for Borel equivalence relations E and F on Polish spaces.

More examples of Suslin orders

- Countable subsets of $\mathbb{R}$ which are linearly independent over $\mathbb{Q}$ (adding a Hamel basis).
- Countable subsets of $\mathbb{C}$ which are algebraically independent over $\mathbb{Q}$ (adding a transcendence basis).
- Countable partial homomorphisms from $(\mathbb{R}, +)$ to itself (adding a discontinuous homomorphism).
- Countable almost disjoint families (of various kind), under containment.
- Disjoint pairs $(a, b) \in [\mathbb{R}]^{<\omega} \times [\mathbb{R}]^{\omega}$, under containment (forces $\neg \mathrm{CC}_{\mathbb{R}}$)

Even more examples

- Countable models of some fixed first order theory T, with domain a set of reals or a set of equivalence classes, under elementary extension.
- Countable assignments of models of some first-order theory (e.g., linear orders, tournaments, acyclic graphs) to E-classes, for some Borel equivalence relation E.
- Countable acyclic subsets of a Borel graph on a Polish space.
- Countable partial ω-colorings of a Borel graph on a Polish space (or k-colorings, for some fixed $k \in \omega$).

Virtual conditions (I)

Let $\langle P, \leq \rangle$ be a Suslin preorder on a Polish space X.

We do not require $\leq$ to be antisymmetric, so different elements of P can represent the same condition in the separative quotient.

So there is an equivalence relation associated to $\leq$.

The corresponding pins are called virtual conditions.

Virtual conditions (II)

A *virtual condition* for a Suslin order $\langle P, \leq \rangle$ is a pair (Q, τ) such that

- Q is a partial order,
- τ is a Q-name for an element of P, and
- τ realizes to an equivalent P-condition in every V-generic Q-extension.

The last item above is the same as : $Q \times Q$ forces that

$$\tau_{g_0} \leq \tau_{g_1} \wedge \tau_{g_1} \leq \tau_{g_0},$$

where g_0 and g_1 are the left and right filters.

Conditions are trivially virtual

If p is a P-condition and Q is any partial order, $(Q, \check{p})$ is a virtual condition.

Nontrivial examples

If P is the partial order of countably generated filters, then

$$(\mathrm{Col}(\omega, F), \dot{g})$$

is a virtual condition, whenever F is a filter on ω (and $\dot{g}$ is a name for the generic enumeration of F).

If P is the partial order of countable partial selectors for a Borel equivalence relation E, then

$$(\mathrm{Col}(\omega, S), \dot{g})$$

is a virtual condition whenever S is a partial selector.

Virtual conditions and ground model conditions

If (Q, τ) and (R, σ) are virtual conditions, then every condition in $Q \times R$ decides the statement

$$\tau \leq \sigma.$$

Similarly, every condition decides whether τ and σ are compatible or not.

Restricing to trivial virtual condtions, it makes sense then to talk about virtual conditions (Q, τ) below (or incompatible with) a given ground model condition p.

Equivalence of virtual conditions

Virtual P-conditions (Q, τ) and (R, σ) are equivalent if

$$Q \times R$$

forces that τ and σ are equivalent conditions in P.

Balanced conditions

Given a Suslin forcing P, a *balanced condition* for P is a virtual condition (Q, τ) such that, for any two V-generic filters

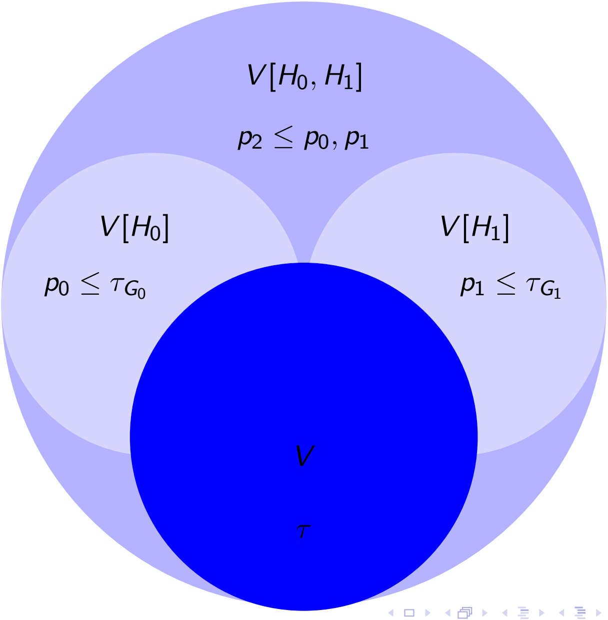
$$G_0, G_1 \subseteq Q$$

existing respectively in mutually generic extensions $V[H_0]$ and $V[H_1]$, and any two conditions

$$p_0 \leq \tau_{G_0} \text{ and } p_1 \leq \tau_{G_1}$$

in $V[H_0]$ and $V[H_1]$ respectively, p_0 and p_1 are compatible in $V[H_0, H_1]$.

Balance



The balanced antichain

If (Q_0, τ_0) and (Q_1, τ_1) are balanced virtual conditions, then they are either equivalent or incompatible.

Otherwise, if $Q_0 \times Q_1$ does not force that $\tau_0 \leq \tau_1$, say, then there are $Q_0 \times Q_1$-names for two strengthenings of the left-realization of τ_0, one stronger than the right realization of τ_1 and one incompatible with it.

This would contradict balance.

Virtual $\mathcal{P}(\omega)/\mathrm{Fin}$ conditions

For the partial order P of countably generated filters ($\mathcal{P}(\omega)/\mathrm{Fin}$), the balanced pairs are (up to equivalence) the pairs

$$(\mathrm{Col}(\omega, U), \dot{g})$$

where U is a ultrafilter on ω and $\dot{g}$ is a name for an enumeration of U in ordertype ω.

Ultrafilters are balanced

Proof idea : If U is an ultrafilter on ω, the union of two mutually generic filters containing U has the finite intersection property, since any name for a member of either of these filters must have U-many possible members below each condition.

Sketch

Let R_0 and R_1 be partial orders, and for each $i \in 2$ let

- $\dot{G}_i$ be an R_i-name for a V-generic filter for $\mathrm{Col}(\omega, U)$
- ρ_i be an R_i-name for a P-condition below the generic enumeration of U given by the realization of $\dot{G}_i$
- σ_i be an R_i-name for a finite intersection of elements of the range of ρ_i
- r_i be a condition in R_i
- A_i be the set of $n \in \omega$ such that $r_i \not\Vdash n \notin \sigma_i$

(continued)

Each A_i must be in U, since its complement can't be, so

$$A_0 \cap A_1 \in U.$$

For any $m \in \omega$, then, one can find $n \in \omega \setminus m$, $r_0' \leq r_0$ and $r_1' \leq r_1$ such that $r_0' \Vdash n \in \sigma_0$ and $r_1' \Vdash n \in \sigma_1$.

If (H_0, H_1) is V-generic for $R_0 \times R_1$, then,

$$\sigma_{0,H_0} \cap \sigma_{1,H_1}$$

will be infinite (for each such pair σ_0, σ_1).

A condition listing the union of the ranges of ρ_0 and ρ_1 will then be below both of them.

Balanced conditions are equivalent to ultrafilters

Conversely, suppose that (Q, τ) is a virtual balanced conditon for P.

For each $A \subseteq \omega$ in the ground model, 1_Q must decide whether or not A is in the filter generated by the range of the realization of τ, since otherwise there are two incompatible generic strengthenings of τ.

Let U be the set of $A \subseteq \omega$ such that 1_Q forces A to be in this filter.

Then $(\mathrm{Col}(\omega, U), \dot{g})$ is a balanced virtual condition which is compatible with (Q, τ), and therefore equal to it.

More examples of balanced conditions

- For the partial order of countable partial selectors for a pinned equivalence relation, the balanced pairs are the $\mathrm{Col}(\omega, 2^{\aleph_0})$-names for enumerations of the total selectors.
- For the partial order of countable partial functions from X to 2 (for X a Polish space), the balanced pairs are the $\mathrm{Col}(\omega, 2^{\aleph_0})$-names for (codes for) total functions from X to 2 in V.
- For the partial order of countable linearly independent subsets of $\mathbb{R}$ over $\mathbb{Q}$, the balanced pairs are the $\mathrm{Col}(\omega, 2^{\aleph_0})$-names for enumerations of Hamel bases.

Even more examples

- If E is a Borel equivalence relation, and P is the partial order of countable partial tournaments on the E-classes, the balanced conditions are classified by the total tournaments on the virtual E-classes.

- For the partial order of disjoint finite/countable pairs of reals, the balanced conditions are characterized by finite sets and their complements.

- For the partial order of countable partial injections from the F-classes to the E-classes, the balanced conditions are the injections from the virtual F-classes to the virtual E-classes (the existence of which may be non-absolute).

Question

Can a Suslin order $\langle P, \leq \rangle$ have a proper class of balanced virtual conditions?

Since a Borel equivalence relation has less than $\beth_{\omega_1}$ many virtual classes, $\langle P, \leq \rangle$ has less than $\beth_{\omega_1}$ many virtual conditions if P and $\leq$ are Borel.

Balanced partial orders

A Suslin order is said to be *balanced* if (provably, in ZFC) below each condition there is a balanced virtual condition.

Question : Is balance absolute between V and some identifiable rank initial segment of V?

Balance is not in general absolute between V and its forcing extensions. In particular, there are Suslin orders which are balanced if and only if CH holds.

We say that P is *cofinally balanced below* κ if every partial order in V_κ is regularly embedded in one forcing that P is balanced in V_κ.

Henle-Mathias-Woodin for cofinally balanced partial orders

Theorem. If

- P is a Suslin order, cofinally balanced below a strongly inaccessible cardinal κ (in V),
- α is an ordinal,
- W is a Solovay model for κ and
- $G \subseteq P$ is W-generic,

then

$$W^\alpha \cap W[G] \subseteq W.$$

We sketch a proof the weaker statement $\mathcal{P}(\alpha) \cap W[G] \subseteq W$, for $\alpha < \kappa$.

Proof sketch

Suppose that, in W, some condition $p \in P$ forces some P-name σ to represent a subset of α. In the Levy extension, σ is definable from some $z \subseteq \omega$ and an element of V.

Let $V[K]$ be a forcing extension of V contained in W such that $p, z \in V[K]$ and such that P is balanced in $V[K]$. Let $(Q, \tau) \in V_\kappa[K]$ be a balanced virtual condition below p.

Fix $\beta < \alpha$ and suppose there exist $Q \times \mathrm{Col}(\omega, {<}\kappa)$-names ρ_0, ρ_1 in $V[K]$ for P-conditions below the realization of τ forcing different truth values (in $W[G]$) to the statement $\check{\beta} \in \sigma$.

Since (Q, τ) is balanced, there are mutually generic extensions $V[K][H_0]$ and $V[K][H_1]$ in which the realizations of ρ_0 and ρ_1 are compatible, giving a contradiction.

Consequences of balance

If P is cofinally balanced then the following hold in $W[G]$:

- every wellordered sequence of elements of W is in W;
- if E and F are Borel equivalence relations such that E is pinned and F is unpinned, then the F-classes do not inject into the E-classes;
- uniformization fails for sets whose cross-sections are equivalence classes of a fixed unpinned Borel equivalence relation;
- there are no infinite MAD families on ω;
- there are no unbounded linearly ordered subsets of $(\omega^\omega, \leq^*)$ or the Turing degrees.

Products

Balance is preserved under countable support products.

So : the conjunction of all the statements forceable by balanced forcing does not imply the negation of any of the statements on the previous slide.