An unexpected connection between a prisoners-and-hats puzzle and set theory

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- Soon became practically a matter of folklore (assuming that this is indeed the first appearance), often also in the version with prisoners and the warden instead of sages and the king (which will be the terminology adopted on further slides).
- Strategy that saves *all of them but eventually one:* the last in line says "white" or "black" depending on the parity of the number of black hats he sees in front of him. Then everybody else can deduce (based on this answer and all the other answers he hears before his turn) his color.

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  - The strategy works even if all the prisoners are deaf (as nobody's guess depends on what happened before him).
  - The strategy works with any number of colors (not necessarily finite, not necessarily countable).

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- Also can be adapted to work with any number of colors.

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Theorem (Hardin & Taylor, 2008)

It is consistent with ZF that, for every possible strategy, there exists an arrangement of hats from ${}^{\omega}2$ *for which the number of prisoners guessing incorrectly is infinite.*

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A warden in a prison takes 3 *prisoners to the yard and puts a hat on each of them. Each hat has an integer number written on it. The prisoners have to choose a finite set of integers, independently of each other, so that a set chosen by at least one them contains the number from that prisoner's hat. Can they achieve the goal?*

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- Solution: each prisoner picks the range between the two numbers he sees.

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Theorem

A strategy exists if and only if CH holds.

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- For $\xi_1, \xi_2 \in \omega_1$, with $\xi_1 < \xi_2$, a prisoner who sees ξ_1 and ξ_2 should choose

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- If $f_\gamma(\alpha) < f_\gamma(\beta)$, then $f_\gamma(\alpha) \in [0, f_\gamma(\beta)]$, which means that the prisoner with α will fulfill the aim. Otherwise, it is analogous for the prisoner with β.

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- This gives $n \in S(t, x)$ for all $n \in \omega$, contradiction!

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  - Let κ be an infinite cardinal. If the hats contain elements of some set of cardinality κ^+, while each prisoner is allowed to choose set of cardinality κ, then a strategy always exists.
  - However, if the hats contain elements of some set of cardinality λ where $\lambda > \kappa^+$, then a strategy does not exist.

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