Failure of an higher analogue of Mho

Novi-Sad conference
In Set theory and general topology

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Outline

Moore's axiom mho

Definition ($\mho$)

There is a sequence $\langle h_\delta : \delta \to \omega \mid \delta < \omega_1 \rangle$ such that for all $\delta < \omega_1$, h_δ is a continuous map from δ into ω such that, for every club $E \subseteq \omega_1$ there is $\delta \in E$ such that $h_\delta[E] = \omega$.

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Remark

Continuity here means that there exists an ω-cofinal subset $C_\delta \subseteq \delta$ such that $h_\delta(\alpha) = h_\delta(\min(C_\delta \setminus \alpha))$ for every $\alpha < \delta$.

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This is in contrast with his theorem that PFA implies that the class of Aronszajn lines admits a basis consisting of a Countryman line $\mathbb{C}$ and its dual $\mathbb{C}^*$.

A generalization

Hereby we consider a natural generalization of Moore's principle $\mho$.

Definition

For a stationary subset S of a regular uncountable cardinal κ, and for a cardinal $\theta < \kappa$, $\mho(S, \theta)$ asserts the existence of a sequence $\langle (h_\delta, C_\delta) \mid \delta \in S \rangle$ such that:

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- For every $\delta \in S$, h_δ is a function from δ to θ;
- For every $\delta \in S$, C_δ is a club in δ of order-type $\mathrm{cf}(\delta)$, and for every $\alpha < \delta$, $h_\delta(\alpha) = h_\delta(\min(C_\delta \setminus \alpha))$;

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- For every cofinal $A \subseteq \kappa$, there is a $\delta \in S$ such that for every $\tau < \theta$,

$$\sup\{\alpha \in A \cap \delta \mid \min(C_\delta \setminus \alpha) \in S \;\&\; h_\delta(\alpha) = \tau\} = \delta.$$

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Note that $\mho(\omega_1, \omega)$ coincides with $\mho$.

Moving one cardinal up

Question

Concentrating on $S^2_1 := \{\delta < \omega_2 \mid \mathrm{cf}(\delta) = \omega_1\}$, *for what* $\theta < \omega_2$ *do* $\mho(S^2_1, \theta)$ *hold?*

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Theorem (F., 2024)

Assuming the consistency of a supercompact cardinal and an inaccessible cardinal above it, it is consistent that $\mho(S^2_1, \omega_1)$ *fails.*

The framework

In order to accomplish our goal, we start with κ a supercompact cardinal and λ some inaccessible above it.

We devise a finite support 'iteration' $\langle \mathbb{O}_\alpha \mid \alpha < \lambda \rangle$ Using virtual models of two-types (countable and uncountable) which will allow us to show $\mathbb{O}_\alpha$ is proper and κ-proper. In the final model, $\aleph_1$ will be preserved and κ will become the new $\aleph_2$.

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Moreover, of course at stage α given a candidate for $\mho(S^2_1, \omega_1)$, forcing with $\mathbb{O}_\alpha$ should eliminate such candidate.
Thus, let us first describe what kind of typical candidates we should consider.

Description of a candidate

A candidate for $\mho(S^2_1, \omega_1)$ is a sequence $\langle (h_\delta, C_\delta) \mid \delta \in S^2_1 \rangle$ such that for every $\delta \in S^2_1$:

- C_δ is a club in δ of order-type ω_1;
- h_δ is a map from δ to ω_1;
- for every $\beta < \delta$, $h_\delta(\beta) = h_\delta(\min(C_\delta \setminus \beta))$;
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Note that we may also assume the following:

- $\min(C_\delta) = 0$ for every $\delta \in S^2_1$;
- if $\delta \in \mathrm{acc}^+(S^2_1)$, then $\mathrm{nacc}(C_\delta) \subseteq \{0\} \cup S^2_1$.

Remark. nacc stands for non-accumulation points

The task ahead

Our task splits into three main parts:

1. Finding a definition for an iterand $\mathbb{O}_\alpha$ that annihilates a given candidate $\langle (h^\alpha_\delta, C^\alpha_\delta) \mid \delta \in S^2_1 \rangle$;

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1. Finding a definition for an iterand $\mathbb{O}_\alpha$ that annihilates a given candidate $\langle (h^\alpha_\delta, C^\alpha_\delta) \mid \delta \in S^2_1 \rangle$;
2. Securing (two types of) properness of $\mathbb{O}_\alpha$;
   i.e. for every $\mathrm{M} \prec H_\chi$ for large enough cardinal χ, given $p \in \mathrm{M} \cap \mathbb{O}_\alpha$ and a dense set D, there exist $p_{\mathrm{M}} \leq p$ such that any $p' \leq p_{\mathrm{M}}$, there exists $q \in D \cap \mathrm{M}$ such that q, p' are compatible.

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3. Repeating the process to take care of all candidates.

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The forcing $\mathbb{O}$ consists of conditions $p := \langle \tau_{\delta,p} \mid \delta \in D_p \cap S^2_1 \rangle$ such that the following hold:

(1) D_p is a subset of $S^2_1 \cup S^2_0$;

(2) For every $\delta \in D_p$ of cof ω_1, $h_\delta[\Delta_1(D_p)] \cap \{\tau_\delta\} = \emptyset$.
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τ_δ plays the role of the prohibited color. Therefore once we add it by some condition, we must respect it. Thus, we assert $p \leq q$ iff:

- $D_p \supseteq D_q$;
- For all $\delta \in \Delta_1(D_q)$, $\tau_{\delta,q} = \tau_{\delta,p}$.

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Unsurprisingly, there is no reason for the poset to be proper. Moreover, for an $\mathbb{O}$-generic G the set $D_G := \bigcup_{p \in G} D_p$ might not be closed. Therefore, one must add protections to ensure properness and closeness.

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Since we are interested in not collapsing three cardinals, we use two kinds of models as side condition, this was introduced by I. Neeman in [Nee14], based on an idea of Todorčević in [Tc85] to add elementary models as side conditions to ensure properness.

In the same paper, Neeman found a way to make the generic set of the models closed. This was done by adding a "decoration" to each model.

Using two-type of models as side conditions

Up to this point, our forcing $\mathbb{O}$ should consist of conditions $p := \langle \mathcal{M}_p, d_p, \langle \tau_{\delta,p} \mid \delta \in \Delta_1(D_p) \rangle \rangle$ where:

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(4) For every $\delta \in D_p$ of cof ω_1, $h_\delta[\Delta_1(D_p)] \cap \{\tau_\delta\} = \emptyset$.

The countable properness

To establish our iterand $\mathbb{O}$ is countably-proper, we are looking at the following setup:
For some large enough cardinal χ, let $\mathrm{M} \prec H_\chi$ be countable with $\mathbb{O} \in \mathrm{M}$. Let p be a condition with $\mathrm{M} \cap H_{\omega_2} \in \mathcal{M}_p$ and $q \in D \cap \mathrm{M}$ for some dense set $D \in \mathrm{M}$ (which is extending some residue of p in M).

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But it raises the following obstructive configuration:

Obstruction

Suppose q is such that,

$$\max(\Delta_1(\mathcal{M}_q)) > \max(\Delta_1(\mathcal{M}_p) \cap \mathsf{M})$$

In this case, For K $\in$-above M, we have to assign a color $\tau_{\delta_{\mathsf{K}}}$ which takes into consideration this extra elements in advance.

The solution

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We add a fifth demand to the poset:

(5) For every uncountable K in $\mathcal{M}_p$ and countable M $\in$-below K of the right form for K in p, $h_{\delta_K}[M] \cap \{\tau_{\delta_K}\} = \emptyset$.

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Remark

We restrict ourselves to specific 'of the right form' models in order to define the notion of a residue in a sound way. Of course the definition of 'right form' also respects the structure of two-type $\in$-chain of models.

Uncountable properness

Now let us try to establish the uncountable properness. The setup in this case is similar: Let χ be some large cardinal, $\mathrm{M} \prec H_\chi$ uncountable with $\mathbb{O} \in \mathrm{M}$. Suppose condition p with $\mathrm{M} \cap H_{\omega_2} \in \mathcal{M}_p$ and $q \in D \cap \mathrm{M}$ for some dense set $D \in \mathrm{M}$ (which is extending some residue of p in M).

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Remark

Note that up to this point, our forcing was actually strongly proper.

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which is strictly greater then ω_1.
Thus, by choosing q more carefully, we get that $\Delta_1(\mathcal{M}_q) \cap C_\delta = \emptyset$ for all $\delta \in \Delta_1(D_p) \setminus \mathrm{M}$. So the forcing is not strongly uncountably-proper, but merely uncountably-proper.

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Unfortunately, we again run into trouble since there is no way to guarantee for example that:

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<u>The solution:</u> Demand Clause (5) to be referred only when $\delta_{\mathsf{P}} \in \mathrm{acc}(C_{\delta_{\mathsf{M}}})$. In our case, as all $\delta_{\mathsf{P}} \in \Delta(\mathcal{M}_q) \setminus \Delta(\mathcal{M}_p)$ are chosen to be outside $C_{\delta_{\mathsf{K}}}$ for all $\delta_{\mathsf{K}} \in \Delta_1(\mathcal{M}_p)$, the situation described above is successfully avoided.

Appropriate conditions

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Lemma

In our context: if q is appropriate for M in p, then for any uncountable $\mathsf{K} \in \mathcal{M}_p$ and countable $\mathsf{Q} \in \mathcal{M}_q$ such that $\mathsf{Q} \notin \mathcal{M}_p$, then one of the following holds:

- $\delta_{\mathsf{Q}} \notin \mathrm{acc}(C_{\delta_{\mathsf{K}}})$;
- *There exists $\mathsf{P} \in \mathcal{M}_p$ which is $\in$-above Q and $\delta_{\mathsf{P}} \in \mathrm{acc}(C_{\delta_{\mathsf{K}}})$.*

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We iteratively add a tower of λ many clubs in κ, eventually collapsing κ to $\aleph_2$, λ to $\aleph_3$. Moreover, at each stage α we ensure that a potential candidate for a witness to $\mho(S^2_1, \omega_1)$ is destroyed by one of the clubs in our tower so far.

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A condition in an 'iterand' of our forcing is a triple of finite sets $\langle \mathcal{M}_p, d_p, F_p \rangle$, where the pair $\langle \mathcal{M}_p, d_p \rangle$ is a condition in the Velickovic-Mohammadpour poset from [MV21]. And F_p is a variation of the working part of the iterand we have just described.


THANKS FOR LISTENING
ANY QUESTIONS?
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