Concentrated sets and γ-sets in the Miller model

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Notation:

- With a *space* we mean a subspace of reals.
- A space is called *totally imperfect* if it contains no copy of the Cantor space 2^ω.
- $\omega^{\uparrow\omega}$: the space of all stricly increasing functions from ω to ω.
- $\omega^\omega \cong \omega^{\uparrow\omega} \cong [\omega]^\omega$.
- $[\omega]^{<\omega}$ are the rationals.

Miller model

Definition

$T \subseteq \omega^{<\omega}$ is a Miller tree if T is closed under initial segments; and for every $t \in T$ there is $s \supseteq t$ such that $s^\frown n \in T$ for infinitely many $n \in \omega$.

- Miller forcing: $\mathbb{M} = \{T \subseteq \omega^{<\omega} : T \text{ is a Miller tree}\}$ with $\leq := \subseteq$.
- $\mathbb{M}_{\omega_2}$: the countable support iteration (c.s.i.) of Miller forcing of length ω_2.

Miller model: Forcing with $\mathbb{M}_{\omega_2}$ over a model of CH.

Concentrated sets and K-Lusin sets

Definition

$X \subseteq 2^\omega$ is *concentrated* on $A \subseteq 2^\omega$ with $|A| = \omega$ if for any open $U \supseteq A$: $|X \setminus U| \leq \omega$. Moreover, we call X *concentrated* if $A \subseteq X$.
$X \subseteq \omega^\omega$ is *K-Lusin* if $|X \cap K| \leq \omega$ for all compact sets $K \subseteq \omega^\omega$.

Observation

For $\kappa > \omega$, there is a concentrated set of size κ iff there is a K-Lusin set of size κ.

Selection Principles

- An open cover is an *ω-cover* if X is not an element of it and every finite subset of X is contained in some element of the cover.
- An open cover is a *γ-cover* if it is infinite and every point is in all but finitely many elements of the cover.

O, Ω, Γ the families of all open covers, ω-covers, γ-covers of X

Example

Let $\mathcal{U} = \{\, A \subseteq \mathbb{R} : A \text{ is open, bounded and } \mu(A) < \frac{1}{\operatorname{diam}(A)} \,\}$, where μ is the Lebesgue measure. Then $\mathcal{U}$ is an ω-cover, but has no subcover that is a γ-cover.

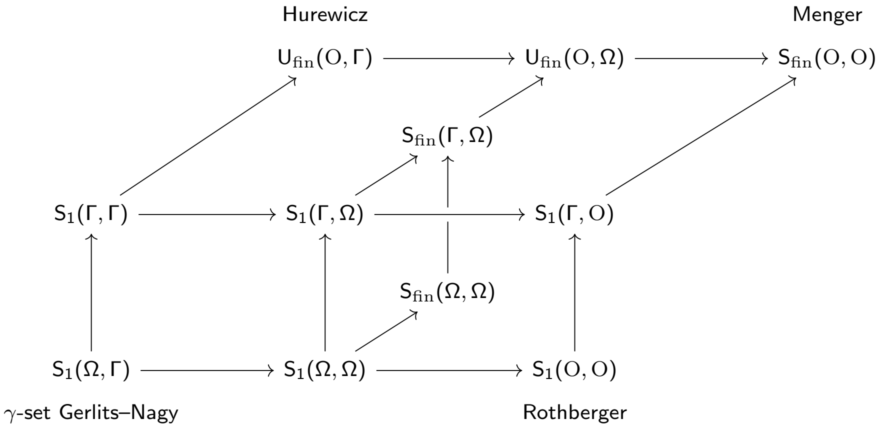
Menger spaces ($\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$)**:** For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is an open cover of X.

Hurewicz spaces ($\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$)**:** For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X, i.e. for all $x \in X$: $|\{ n \in \omega : x \notin \bigcup \mathcal{V}_n \}| < \omega$.

Rothberger spaces ($\mathsf{S}_1(\mathrm{O},\mathrm{O})$)**:**
For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there is a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}_n$, and $\{ U_n : n \in \omega \}$ is an open cover of X.

γ**-sets** ($\mathsf{S}_1(\Omega,\Gamma)$)**:**
For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of ω-covers there are sets $U_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{ U_n : n \in \omega \}$ is a γ-cover of X.

Scheepers diagram



The Scheepers Diagram illustrating the connections among the selection principles, excluding trivial ones or those that are equivalent in ZFC.

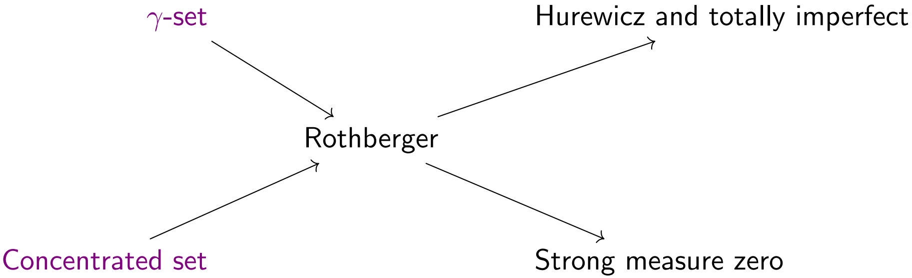
In the Miller model:

Some properties:

- $\mathfrak{c} = \mathfrak{d} = \omega_2$
- $\mathfrak{b} = \mathfrak{p} = \omega_1$
- $\mathfrak{u} < \mathfrak{g}$
- There are totally imperfect Menger subspaces of reals of size $\mathfrak{c}$.
- The ground-model reals are unbounded and the Miller reals are unbounded.

Theorem (Zdomskyy, 2005)

In the Miller model, every Rothberger space is Hurewicz.



Conjecture (Bartoszyński, Halbeisen, 2003)

In the Miller model, there are K-Lusin sets of size $\mathfrak{c}$.

Our Goal:

This is not only false for K-Lusin sets (equivalently concentrated sets), but for all the properties above.

We have a partial realization, namely for Concentrated sets and γ-sets.

Forcing Combinatorics

$[f < g] := \{ n \in \omega : f(n) < g(n) \}$.

Definition

Let $h \in \omega^{\uparrow\omega}$.
$f \in \omega^{\uparrow\omega}$ is *h-unbounded* over a set N if
$\{ n \in \omega : [h(n), h(n+1)) \subseteq [x < f] \}$ infinite for all $x \in \omega^\omega \cap N$.

Definition

A poset $\mathbb{P}$ is *mild* if for elementary submodel $N \ni \mathbb{P}$ of $H(\theta)$ with sufficiently large θ, if $f \in \omega^{\uparrow\omega}$ is h-unbounded over N for some $h \in \omega^{\uparrow\omega}$, then for every $p \in \mathbb{P} \cap N$ there is an $(N, \mathbb{P})$-generic condition $q \leq p$ such that

$$q \Vdash f \text{ is } h\text{-unbounded over } N[\Gamma].$$

Forcing Combinatorics

Lemma

If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *is a c.s.i. of mild posets, then* $\mathbb{P}_\delta$ *is also mild.*

Lemma

$\mathbb{M}_{\omega_2}$ *is mild.*

Example

Cohen forcing is mild and Laver forcing is not mild.

Forcing Combinatorics

Lemma

Let $\mathbb{P}$ be a mild poset and let G be a $\mathbb{P}$-generic filter over V. If $x \in (\omega^\omega)^{V[G]}$, and $\psi : (\omega^{\uparrow\omega})^V \to (\omega^{\uparrow\omega})^V$ is a function which is an element of V, then there exists an element $f \in \omega^{\uparrow\omega} \cap V$ such that the set

$$\{ n \in \omega : [\psi(f)(n), \psi(f)(n+1)) \subseteq [x < f] \}$$

is infinite.

In particular, the above holds for $\mathbb{M}_{\omega_2}$.

Proof.

Let $\dot{x}$ be a name for x. We work in V:

- Let $p \in \mathbb{P}$.
- Pick an elementary submodel N such that $p, \dot{x} \in N$.
- Fix $f \in \omega^{\uparrow\omega}$ such that $z <^* f$ for all $z \in \omega^\omega \cap N$.
- f is $\psi(f)$-unbounded over N.
- Let $q \leq p$ be an $(N, \mathbb{P})$-generic condition with $q \Vdash f$ is $\psi(f)$-unbounded over $N[\Gamma]$.
- In particular, $q \Vdash \{ n \in \omega : [\psi(f)(n), \psi(f)(n+1)) \subseteq [\dot{x} < f] \}$ is infinite.

□

Concentrated sets in the Miller model

Theorem (H., Szewczak, Zdomskyy, 2023)

In the Miller model, there is no K-Lusin set in $\omega^{\uparrow\omega}$ of size $\mathfrak{c}$. Equivalently, in this model there is no concentrated set of size $\mathfrak{c}$.

Proof (Idea).

Assume there is such X. Since Rothberger implies being Hurewicz, we can find $\psi : \omega^{\uparrow\omega} \to \omega^{\uparrow\omega}$ such that $[\psi(f)(n), \psi(f)(n+1)) \cap [x \geq f] \neq \emptyset$ for all but finitely many n if $x \not\leq^* f$.
By mildness (Lemma above) and an intermediate submodel argument, we can find a contradiction. □

Concentrated sets in the Miller model

Lemma (Folklore)

In the Miller model, there are concentrated sets of reals of size ω_1*.*

Proof.

Since $\mathfrak{b} = \omega_1$, any $\mathfrak{b}$-scale $\{b_\beta : \beta < \mathfrak{b}\}$ is concentrated on a copy Q of the rationals. Hence $\{b_\beta : \beta < \mathfrak{b}\} \cup Q$ is concentrated. □

γ-sets in the Miller model

Recall X is a γ-set if for every sequence of ω-covers $\langle \mathcal{U}_n : n \in \omega \rangle$ there exists a γ-cover $\{U_n :\in \omega\}$ with $U_n \in \mathcal{U}_n$.

Theorem (Orenshtein, Tsaban, 2011)

If $\mathfrak{p} = \mathfrak{b}$, then there is a γ-set of cardinality $\mathfrak{p}$.
In particular, in the Miller model there are γ-sets of reals of size ω_1.

Semifilters

$S \subseteq [\omega]^\omega$ is a *semifilter* if: $\forall y \in S\ \forall x \in \mathcal{P}(\omega)\ (y \subseteq^* x \Rightarrow x \in S)$.
If $\varphi : \omega \to \omega$ is a finite-to-one function, we denote
$\varphi(S) := \{ x \in \mathcal{P}(\omega) : \varphi^{-1}[x] \in S \} = \{ x \in \mathcal{P}(\omega) : \exists\, y \in S(\varphi[y] \subseteq x) \}$.
Note, a filter $\mathcal{F}$ is a semifilter iff $\mathfrak{Fr} := \{x \in \mathcal{P}(\omega) : \omega \setminus x \text{ finite}\} \subseteq \mathcal{F}$.

Definition

The *semifilter trichotomy* is the statement that for every semifilter S exatly one of the following assertions holds: Fix an ultrafilter $\mathcal{U}$.

1. There is a monotone increasing surjection $\varphi : \omega \to \omega$ such that $\varphi(S) = \mathfrak{Fr}$.
2. There is a monotone increasing surjection $\varphi : \omega \to \omega$ such that $\varphi(S) = \varphi(\mathcal{U})$.
3. There is a monotone increasing surjection $\varphi : \omega \to \omega$ such that $\varphi(S) = [\omega]^\omega$.

Semifilter tools

In the Miller model: $\omega_1 = \mathfrak{u} < \mathfrak{g} = \omega_2$.

Theorem (Blass, Laflamme)

The semifilter trichotomy holds iff $\mathfrak{u} < \mathfrak{g}$.

Lemma

In the Miller model, suppose that $X \supseteq [\omega]^{<\omega}$ is a γ-set. Then $X \setminus [\omega]^{<\omega}$ is bounded by ω_1-many elements of $[\omega]^\omega$.

Weakly G_{ω_1}-concentrated

R is G_{ω_1} if $R = \bigcap_{i<\omega_1} O_i$, with O_i open.
X is called *weakly G_{ω_1}-concentrated* if:
For every collection $\mathcal{C} \subseteq [X]^\omega$ which is cofinal with respect to inclusion, and for every $R : \mathcal{C} \to \mathcal{P}(X)$ assigning to each $Q \in \mathcal{C}$ a G_{ω_1}-set $R(Q)$ containing Q, there exists $\mathcal{C}_1 \in [\mathcal{C}]^{\omega_1}$ such that $X \subseteq \bigcup_{Q\in\mathcal{C}_1} R(Q)$.

Theorem (Zdomskyy, 2018)

in the Miller model, each Menger subspace of $\mathcal{P}(\omega)$ is weakly G_{ω_1}-concentrated.

In particular, the same holds for γ-sets.

γ-sets in the Miller model

Theorem (H., Szewczak, Zdomskyy, 2024)

In the Miller model, there are no γ-sets $X \subseteq \mathcal{P}(\omega)$ of size $\mathfrak{c}$.

Proof (Idea).

Let $\mathcal{C} \subseteq [X]^\omega$ be the collection of all dense countably infinite subsets of X.
$\mathcal{C}$ is clearly cofinal in $[X]^\omega$.
By semifilter trichotomy and γ-set property: Any dense $Q \in [X]^\omega$ is a G_{ω_1}-set in X.

- Take $R : \mathcal{C} \to \mathcal{P}(X)$ with $R(Q) = Q$.
- $X \subseteq \bigcup_{Q \in \mathcal{C}_1} R(Q) = \bigcup_{Q \in \mathcal{C}_1} Q$ with $\mathcal{C}_1 \in [C]^{\omega_1}$.
- Thus, $|X| \leq \omega_1$.

□

Open Problems

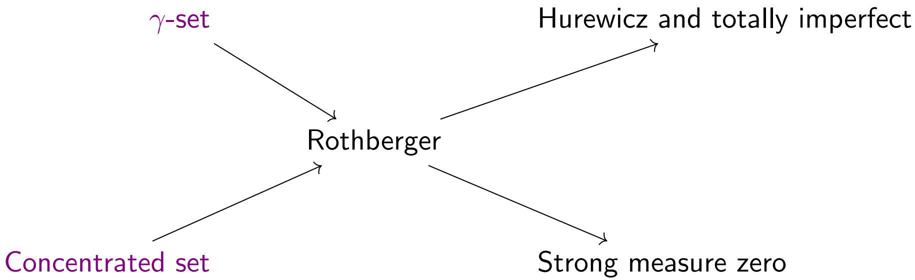
Problem

Is it consistent that there exists a set $X \subseteq [\omega]^\omega$ of size $|X| > \omega_1$ such that for every $f \in \omega^{\uparrow\omega}$

- *X is K-Lusin and*
- *there exists $\psi(f) \in \omega^{\uparrow\omega}$ such that for every $x \in X$, if $x \not\leq^* f$, then*

$$[\psi(f)(n), \psi(f)(n+1)) \cap [x \geq f] \neq \emptyset$$

for all but finitely many $n \in \omega$?



Problem

Is there, in the Miller model,

- *a Rothberger set $X \subseteq 2^\omega$ of size $\mathfrak{c}$?*
- *a totally imperfect Hurewicz set $X \subseteq 2^\omega$ of size $\mathfrak{c}$?*
- *a strong measure zero set $X \subseteq 2^\omega$ of size $\mathfrak{c}$?*

Hvala na pažnji!

Thank you for your attention!