An ordinal rank measuring universality

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Joint work (in progress) with S. Shelah

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Main task

Study a natural rank on models or more abstract mathematical structures that tells us how complicated they are.

References

References

1. W. Kubiś, P. Radecka, *Evolution systems: A framework for studying generic mathematical structures*, https://arxiv.org/abs/2109.12600

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1. W. Kubiś, P. Radecka, *Evolution systems: A framework for studying generic mathematical structures*, `https://arxiv.org/abs/2109.12600`
2. W. Kubiś, S. Shelah, *Analytic colorings*, Ann. Pure Appl. Logic 121 (2003), no. 2-3, 145–161

Evolution systems

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Definition

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category (called the universe), Θ is a fixed $\mathfrak{V}$-object, called the origin, and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows, called transitions.

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The only requirements are:

1. All identities are in $\mathcal{T}$.
2. $h \circ t \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an isomorphism.
3. (Regularity) $t \circ g \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and g is an isomorphism.

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Local smallness

Given an object X, denote by $\mathcal{T}(X)$ the class of all transitions with domain X.
We shall assume that for each X there is a *set* $\mathcal{S}(X) \subseteq \mathcal{T}(X)$ that *covers* $\mathcal{T}(X)$ in the sense that for every $t \in \mathcal{T}(X)$ there is $t' \in \mathcal{S}(X)$ and an isomorphism h satisfying $t = h \circ t'$.

Evolutions

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$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition.
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A path is a finite composition of transitions.

Finite and countable objects

Definition

An object X is finite if there exists a path from the origin Θ to X.

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An object M is countable if there exists an evolution whose colimit is M.

The category of countable objects with *countable paths* will be denoted by $\mathcal{E}^{\sigma}$.

Basic examples

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Example

Let $\mathcal{F}$ be a hereditary class of models of a fixed relational first-order language. We declare Θ to be the trivial structure. Transitions are one-point extensions and isomorphisms. The universe $\mathfrak{V}$ could consist either of all embeddings or all homomorphisms.

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Example

Let $\mathcal{C}$ be a category of profinite first-ordered structures, again in a relational language. Now we declare Θ to be the singleton and we declare $f \colon X \to Y$ to be a transition if it is a continuous quotient epimorphism and there is at most one $y_0 \in Y$ with $|f^{-1}(y_0)| = 2$, while $|f^{-1}(y)| = 1$ for every $y \in Y \setminus \{y_0\}$.

A concrete example

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Example

Fix an integer $k > 0$. Let $\mathfrak{V}$ be the category of graphs, Θ the singleton graph. We declare $t \colon G \to G'$ a transition if it is either an isomorphism or else

$$G' = t[G] \cup \{v\}$$

and v is connected to at most k vertices of $t[G]$.

Abstract Fraïssé theory

Definition

We say that $\mathcal{E}$ has the transition amalgamation property (briefly: TAP) if for every finite object Z, for every transitions $f\colon Z \to X$, $g\colon Z \to Y$ there exist transitions f', g' such that $f' \circ f = g' \circ g$.

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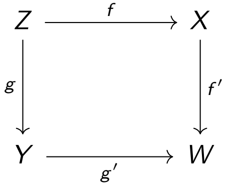

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An evolution $\vec{u}$ has the absorption property if for every n, for every transition $t\colon U_n \to Y$ there are $m > n$ and a path $g\colon Y \to U_m$ such that $f \circ t = u_n^m$.

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$$\begin{array}{ccccccccccccc} \Theta = U_0 & \longrightarrow & U_1 & \longrightarrow & \cdots & \longrightarrow & U_n & \longrightarrow & \cdots & \longrightarrow & U_m & \longrightarrow & \cdots \\ & & & & & & \downarrow & & & & & & \\ & & & & & & Y & & & & & & \end{array}$$

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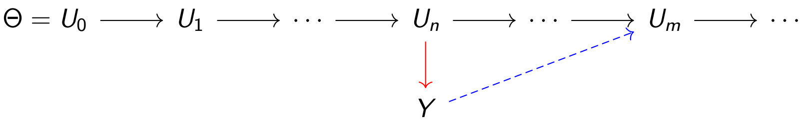

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Theorem

The $\mathcal{E}$-generic object U is $\mathcal{E}^{\mathrm{fin}}$-homogeneous and $\mathcal{E}^{\sigma}$-cofinal.
More precisely:

1. *Given a finite object A and paths $f_i \colon A \to U$, $i = 0, 1$, there is an automorphism $h \colon U \to U$ such that $f_1 = h \circ f_0$.*
2. *For every countable object M there is a countable path $p \colon M \to U$.*

Functors

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Definition

A functor of evolution systems is a functor of their universes that preserves both the origin and the transitions.

The rank of a functor

A transition is nontrivial if it is not an isomorphism.

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Definition

Let $\Phi\colon \mathcal{E} \to \mathcal{E}'$ be a functor of evolution systems. We define the rank $\mathrm{rk}_\Phi(A)$ for each finite object A in $\mathcal{E}$ by the following rules.

(O) $\mathrm{rk}_\Phi(A)$ is either an ordinal or ∞, and we agree that ∞ is above all ordinals.

(R) $\mathrm{rk}_\Phi(A) \geq \alpha + 1$ if and only if for every nontrivial transition $f\colon \Phi A \to Y$ there exists a transition $t\colon A \to B$ such that $\mathrm{rk}_\Phi(B) \geq \alpha$ and Φt is left-isomorphic to f, that is, $f = h \circ \Phi t$ for some isomorphism h.

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Another option is

$$\mathrm{rk}(\Phi) := \sup\{\mathrm{rk}_\Phi(A)\colon A \in \mathrm{Obj}(\mathcal{E}^{\mathrm{fin}})\}.$$

Basic properties

Proposition

Let $\Phi\colon \mathcal{E} \to \mathcal{E}'$ *be a functor of evolution systems. Then*

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for every isomorphic finite objects A, B.

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Proposition

Let $\Phi\colon \mathcal{E} \to \mathcal{E}'$ *be a functor of evolution systems. If* $\mathcal{E}$ *is locally countable, then* rk_Φ *has values in* $\omega_1 \cup \{\infty\}$.

Definition

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Define a new system $\mathcal{E}_M = \langle \mathfrak{V}_M, \mathcal{T}_M, \Theta_M \rangle$ as follows.

(0) The objects of $\mathfrak{V}_M$ are paths from finite objects into M. The arrows are $\mathfrak{V}$-arrows making $\mathfrak{V}_M$ a comma category. Specifically, if $f\colon A \to M$, $g\colon B \to M$ are $\mathfrak{V}_M$-objects, an arrow from f to g is any $\mathfrak{V}$-arrow $k\colon A \to B$ satisfying $f = g \circ k$.

(1) Transitions are the same as in $\mathcal{E}$, now treated as $\mathfrak{V}_M$-arrows.

(2) Θ_M is the unique path from Θ to M.

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There is a canonical functor $\Phi_M\colon \mathcal{E}_M \to \mathcal{E}$, forgetting M.

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Definition

Define $\mathrm{rk}(M) := \mathrm{rk}(\Phi_M)$.

Main results

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Theorem

Assume $\mathcal{E}$ is locally countable, with the TAP, and let U be its generic object. Given a $\mathfrak{V}$-object M, the following properties are equivalent.

(a) $\mathrm{rk}(M) = \infty$.

(b) *There exists a $\mathfrak{V}$-arrow $f: U \to M$.*

A poster with a simplified version can be found at

`https://users.math.cas.cz/kubis/pdfs/ranksEvasPS_ver2.pdf`

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THE END