On generic topological embeddings

Andrzej Kucharski

joint work with W. Kubiś and S. Turek

University of Silesia in Katowice

Novi Sad Conference in Set Theory and General Topology 2024

August 19-22, 2024

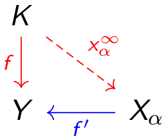
Let κ be a regular cardinal and $\mathfrak{K}$ be a full subcategory of a bigger category $\mathfrak{L}$ such that the following compatibility conditions are satisfied.

(L0) All $\mathfrak{L}$-arrows are epi.

(L1) Every inverse sequence of length κ in $\mathfrak{K}$ has the limit in $\mathfrak{L}$.

(L2) Every $\mathfrak{L}$-object is the limit of an inverse sequence in $\mathfrak{K}$.

(L3) For every inverse sequence $\vec{x}$ in $\mathfrak{K}$ with $K = \lim \vec{x}$ in $\mathfrak{L}$, for every $\mathfrak{K}$-object Y, for every $\mathfrak{L}$-arrow $f \colon K \to Y$ there exist $\alpha < \kappa$ and a $\mathfrak{K}$-arrow $f' \colon X_\alpha \to Y$ such that $f = f' \circ x_\alpha^\infty$.

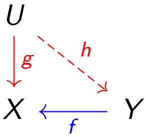


$\mathfrak{K}$-generic

Now, an $\mathfrak{L}$-object U will be called $\mathfrak{K}$-generic if

(G1) $\mathfrak{L}(U, X) \neq \emptyset$ for every $X \in \mathrm{Obj}(\mathfrak{K})$.

(G2) For every $\mathfrak{K}$-arrow $f\colon Y \to X$, for every $\mathfrak{L}$-arrow $g\colon U \to X$ there exists an $\mathfrak{L}$-arrow $h\colon U \to Y$ such that $f \circ h = g$.



The Cantor set $\{0,1\}^\omega$

Let $\mathfrak{Fin}$ be a category of finite nonempty discrete spaces

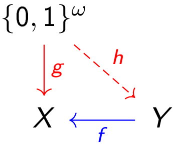
as a full subcategory of the category

$\mathfrak{Comp}$ of compact metric spaces and continuous surjections

.

(G1) $\mathfrak{Comp}(\{0,1\}^\omega, X) \neq \emptyset$ for every $X \in \mathrm{Obj}(\mathfrak{Fin})$.

(G2) For every $\mathfrak{Fin}$-arrow $f\colon Y \to X$, for every $\mathfrak{Comp}$-arrow $g\colon \{0,1\}^\omega \to X$ there exists an $\mathfrak{Comp}$-arrow $h\colon \{0,1\}^\omega \to Y$ such that $f \circ h = g$.



The Cantor set is the $\mathfrak{Fin}$-generic object.

The Čech-Stone remainder ω^*

Let $\mathfrak{Comp}$ be a category of compact metric spaces and continuous surjections
and
let $\mathfrak{L}$ be a category of compact spaces of weight not greater than the continuum, with continuous surjections.

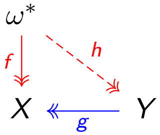
The following results of Parovičenko and Negrepontis imply that ω^* is $\mathfrak{Comp}$-generic.

(G1)Theorem (Parovičenko)

Every compact metric space is a continuous image of ω^*

(G2)Theorem (Negrepontis)

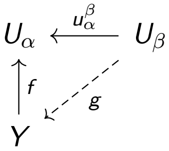
Assuming [CH], ω^* is compact Hausdorff space of weight ω_1 such that for every two continuous surjections $f\colon \omega^* \to X$ and $g\colon Y \to X$ with X and Y compact metric, there exists a continuous surjection $h\colon \omega^* \to Y$ such that $g \circ h = f$



A κ-Fraïssé sequence in $\mathfrak{K}$ is an inverse sequence $\vec{u}$ of regular length κ satisfying the following conditions:

(U) For every object X of $\mathfrak{K}$ there exists $\alpha < \kappa$ such that $\mathfrak{K}(U_\alpha, X) \neq \emptyset$.

(A) For every $\alpha < \kappa$ and for every morphism $f\colon Y \to U_\alpha$, where $Y \in \mathrm{Obj}(\mathfrak{K})$, there exist $\beta \geq \alpha$ and $g\colon U_\beta \to Y$ such that $u_\alpha^\beta = f \circ g$.



Categories of κ-ultrametric spaces

Let γ be an ordinal. A γ-ultrametric (also called an "inverse γ-metric") on a set X is a function $u\colon X \times X \to \gamma + 1$ such that for all $x, y, z \in X$:

(U1) $u(x, y) = \gamma$ if and only if $x = y$,

(U2) $u(y, z) \geq \min\{u(y, x), u(x, z)\}$ (ultrametric triangle law),

(U3) $u(x, y) = u(y, x)$ (symmetry).

Let κ, λ be infinite cardinals.
We define a λ-ultrametric $u \colon \kappa^\lambda \times \kappa^\lambda \to \lambda + 1$ by the formula:

$$u(a, b) = \sup\{\alpha < \lambda \colon a \restriction \alpha = b \restriction \alpha\}$$

for $a, b \in \kappa^\lambda$.
If X is a discrete space, then for any ordinal $\gamma \geq \omega$ there is a natural γ-ultrametric $d \colon X \times X \to \gamma + 1$ on X, namely $d(a, b) = \gamma$ iff $a = b$ and $d(a, b) = 0$ iff $a \neq b$. Since $B_\gamma(x) = \{x\}$ the set $\{x\}$ is open for any $x \in X$. So in this case γ-ultrametric topology is discrete.

A closed ball of radius α and center x is a set of the form

$$B_\alpha(x) = \{y \in X : u(x, y) \geq \alpha\},$$

where $x \in X$ and $\alpha \in \gamma$. Each γ-ultrametric induces a topology whose base is the family of all closed balls of radius less then γ. We will call a space with this topology γ-ultrametric.

We say that a λ-ultrametric space (X, d) of weight not greater than $\kappa^{<\lambda}$ is (λ, κ)-bounded if there is a non-decreasing sequence of ordinals numbers $\{\gamma_\alpha : \alpha < \lambda\} \subset \lambda$ such that $|\{B_\alpha(a) \colon a \in X\}| \leq |\kappa^{\gamma_\alpha}|$ for every $\alpha < \lambda$.

A space κ^ω with ultrametric defined as above is (ω, κ)-bounded.

From here on, we will assume that κ, λ are regular cardinals such that $\lambda \leq \kappa$.

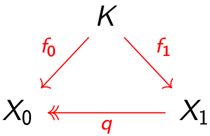
A λ-ultrametric space X is spherically complete if every nonempty chain of closed balls has nonempty intersection.

Theorem (Kubiś, K., Turek)

A topological space X is (λ, κ)-bounded and spherically complete if and only if there exists a non-decreasing sequence of ordinals $\{\gamma_\alpha : \alpha < \lambda\} \subset \lambda$ and exists an inverse sequence $\{X_\alpha, q_\alpha^\beta, \alpha \leq \beta < \lambda\}$ such that

- $X = \varprojlim\{X_\alpha, q_\alpha^\beta, \alpha \leq \beta < \lambda\}$,
- X_α *is a discrete space of cardinality not greater than* $|\kappa^{\gamma_\alpha}|$ *for each* $\alpha < \lambda$,
- $q_\alpha^\beta \colon X_\beta \to X_\alpha$ *is surjection for all* $\alpha \leq \beta < \lambda$.

Fix a λ-ultrametric space (K, u) of weight κ. The objects of $\mathfrak{M}_K$ are uniformly continuous mappings $f\colon K \to X$, where (X, d) is a (λ, κ)-bounded and spherically complete.. Given two $\mathfrak{M}_K$-objects $f_0\colon K \to X_0$, $f_1\colon K \to X_1$, an $\mathfrak{M}_K$-arrow from f_1 to f_0 is a uniformly continuous surjection $q\colon X_1 \twoheadrightarrow X_0$ satisfying $q \circ f_1 = f_0$. The composition in $\mathfrak{M}_K$ is the usual composition of mappings.



Let $\mathfrak{D}_K$ be the full subcategory of $\mathfrak{M}_K$ whose objects are $f\colon K \to X$ such that X is a discrete space of cardinality not greater than $|\kappa^\gamma|$ for some $\gamma < \lambda$.

We say that a function $f\colon K \to X$ is uniformly continuous if

$$\forall_{\epsilon\in\tau}\exists_{\delta\in\kappa}\forall_{a,b\in K}\ u(a,b)\geq\delta \Rightarrow d(f(a),f(b))\geq\epsilon,$$

where $u\colon K\times K\to\kappa+1$ and $d\colon X\times X\to\tau+1$ are ultrametrics; i. e. given $B_\epsilon(f(a))$ there is $\delta\in\kappa$ such that

$$f^{-1}[B_\epsilon(f(a))] = \bigcup\left\{B_\delta(c)\colon c\in f^{-1}[B_\epsilon(f(a))]\right\}.$$

A family of arrows $\mathcal{F}$ is dominating in $\mathfrak{K}$ if it satisfies the following two conditions.

(i) For every $X \in \mathrm{Obj}\,(\mathfrak{K})$ there is $A \in \mathrm{Cod}(\mathcal{F})$ such that $\mathfrak{K}(A, X) \neq \emptyset$.

(ii) Given $A \in \mathrm{Cod}(\mathcal{F})$ and $f \in \mathfrak{K}(Y, A)$ there exist $g \in \mathfrak{K}(B, Y)$ such that $f \circ g \in \mathcal{F}$.

Theorem (Kubiś (2014))

Let κ be an infinite regular cardinal and let $\mathfrak{K}$ be a κ-complete directed category with the amalgamation property. Assume further that $\mathcal{F} \subseteq \mathfrak{K}$ is dominating in $\mathfrak{K}$ and $|\mathcal{F}| \leq \kappa$. Then there exists a continuous Fraïssé sequence of length κ in $\mathfrak{K}$.

Theorem (Kubiś, K.,Turek)

There exists a continuous Fraïssé λ-sequence in $\mathfrak{D}_K$.

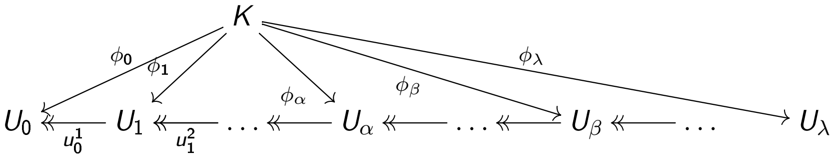
Theorem (Kubiś, K., Turek)

Assume that $\vec{\phi} = (\phi_\alpha \colon \alpha < \lambda)$ *is a continuous Fraïssé* λ*-sequence in* $\mathfrak{D}_K$*, where* $\phi_\alpha \colon K \to U_\alpha$ *for each* $\alpha < \kappa$*. Then*

(1) $\vec{u} = (U_\alpha \colon \alpha < \lambda)$ *is a Fraïssé sequence in the category of discrete spaces of cardinality not greater than* $|\kappa^\gamma|$ *for some* $\gamma < \lambda$ *and surjection.*

(2) *The limit map* $\phi_\lambda \colon K \to \lim \vec{u}$ *has a left inverse, i.e. there is* $r \colon \lim \vec{u} \to K$ *such that* $r \circ \phi_\lambda = \mathrm{id}_K$*.*

(3) *The image* $\phi_\lambda[K] \subseteq U_\lambda = \lim \vec{u}$ *is uniformly nowhere dense.*



We say that a subset A of a λ-ultrametric space (X, d) of weight κ is uniformly nowhere dense if for every $\alpha < \lambda$ there is $\beta > \alpha$ such that

$$\{B_\beta(a)\colon B_\beta(a) \cap A = \emptyset\ , a \in B_\alpha(b)\} \neq \emptyset$$

for every $b \in X$. Note that every uniformly nowhere dense subset of the ultrametric space κ^λ is nowhere dense.

Corollary (Kubiś, K., Turek)

A λ-ultrametric space (K, u) of weight κ can be uniformly embedded into κ^λ as a uniformly nowhere dense subset.

Theorem (Kubiś, K., Turek)

If $\eta\colon K \to \kappa^\lambda$ is uniformly embedded such that $\eta[K]$ is uniformly nowhere dense in the λ-ultrametric space κ^λ, then $\eta\colon K \to \kappa^\lambda$ is $\mathfrak{D}_K$-generic.

Corollary (Kubiś, K., Turek)

Every uniform homeomorphism of uniformly nowhere dense sets in κ^λ can be extended to a uniform auto-homeomorphism of κ^λ.

Corollary (Kubiś, K., Turek)

Every uniformly nowhere dense set in κ^λ is a retract of κ^λ.

A topological space X is *κ-compact* if every open cover of X has a subcover of size strictly less than κ.

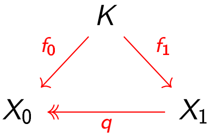
A cardinal κ is *weakly compact* if it is uncountable and has the Ramsey property $\kappa \to (\kappa)^2$, i.e., if $f\colon [\kappa]^2 \to \{0,1\}$ then there are $i \in \{0,1\}$ and $A \in [\kappa]^\kappa$ such that $f \restriction A = \{i\}$

Theorem (Monk, Scott)

A cardinal number κ is weakly compact if and only if the κ-ultrametric 2^κ is κ-compact.

From here on, we will assume that κ is a weakly compact cardinal.

Fix a κ-compact ultrametric space (K, u) of weight not greater than κ, where $u \colon K \times K \to \kappa + 1$.
The objects of $\mathfrak{M}^w_K$ are continuous mappings $f \colon K \to X$, where (X, d) is a κ-ultrametric space of weight not greater than κ. Given two $\mathfrak{M}^w_K$-objects $f_0 \colon K \to X_0$, $f_1 \colon K \to X_1$, a $\mathfrak{M}^w_K$-arrow from f_1 to f_0 is a continuous surjection $q \colon X_1 \twoheadrightarrow X_0$. The composition in $\mathfrak{M}^w_K$ is the usual composition of mappings.



We define $\mathfrak{D}^w_K$ to be the full subcategory of $\mathfrak{M}^w_K$ whose objects are $f \colon K \to X$ such that X is a discrete spaces of cardinality $< \kappa$.

Theorem (Kubiś, K., Turek)

Assume that $\vec{\phi} = (\phi_\alpha \colon \alpha < \kappa)$ *is a continuous Fraïssé* κ*-sequence in* $\mathfrak{D}^w_K$*, where* $\phi_\alpha \colon K \to U_\alpha$ *for each* $\alpha < \kappa$*. Then*

(1) $\vec{u} = (U_\alpha \colon \alpha < \kappa)$ *is a Fraïssé sequence in the category of discrete spaces of cardinality less than* κ*.*

(2) *Then the limit map* $\phi_\kappa \colon K \to \lim \vec{u}$ *has a left inverse, i.e., there is* $r \colon \lim \vec{u} \to K$ *such that* $r \circ \phi_\kappa = \mathrm{id}_K$*.*

(3) *The image* $\phi_\kappa[K] \subseteq U_\kappa = \lim \vec{u}$ *is nowhere dense.*

Corollary (Kubiś, K., Turek)

A κ-compact ultrametric space (K, u) of weight not greater than κ can be embedded into the κ-ultrametric space 2^κ as a nowhere dense subset.

Theorem (Kubiś, K., Turek)

If $\eta\colon K \to 2^\kappa$ is an embedding such that $\eta[K]$ is nowhere dense in the κ-ultrametric space 2^κ, then $\eta\colon K \to 2^\kappa$ is a $\mathfrak{D}_K^w$-generic.

Knaster and Reichbach established the following theorem:

> *If P and K are closed nowhere dense subsets of the Cantor space 2^ω and f is a homeomorphism between P and K, then there exists a homeomorphism between the Cantor space extending f.*

Below we have a counterpart of this theorem for the κ-ultrametric space 2^κ.

Corollary (Kubiś, K.,Turek)

Every homeomorphism of nowhere dense subsets of the κ-ultrametric space 2^κ can be extended to an auto-homeomorphism of 2^κ.

Corollary (Kubiś, K.,Turek)

Every nowhere dense set in the κ-ultrametric space 2^κ is a retract.

Refereces

W. Kubiś, A. Kucharski and S. Turek, *On generic topological embeddings*, preprint, arXiv:2310.05043, (2023)

Thank You for Your attention!