Fraïssé's conjecture and big Ramsey degrees of structures admitting finite monomorphic decomposition

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Coloring the infinite

Theorem [Ramsey 1930]

Let X be a countably infinite set. For any finite coloring of $[X]^n$ there is an infinite $M \subseteq X$ such that $[M]^n$ is monochromatic.

Frank P. Ramsey
(1903–1930)

Coloring the infinite

$\aleph_0 \longrightarrow (\aleph_0)^n_k$

partition calculus
for cardinals

$\omega \longrightarrow (\omega)^n_k$

partition calculus
for ordinals

$\mathbb{N} \longrightarrow (\mathbb{N})^n_k$

partition calculus
for linear orders

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Coloring the infinite

partition calculus
for structures

$\mathcal{S} \longrightarrow (\mathcal{S})^{\mathcal{A}}_k$

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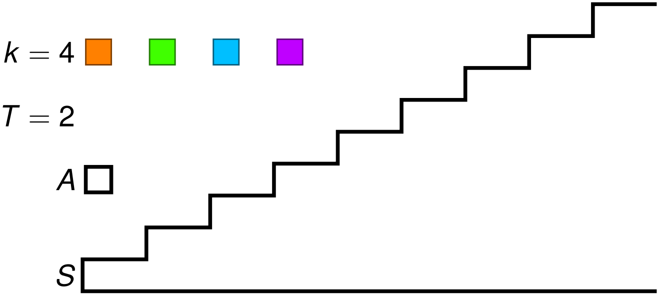
The infinite is usually not monochromatic

Definition [Kechris, Pestov, Todorčević]

Let S be a structure and A a finite substructure of S.

- The big Ramsey degree of A in S is the least $t \in T$ such that for every finite coloring $\chi : \mathrm{Emb}(A, S) \to k$ there is an isomorphic copy $C \leqslant S$ such that $|\chi(\mathrm{Emb}(A, C))| \leqslant t$.
- We write $T(A, S) = t$, or $T(A, S) = \infty$ if no such t exists.

The infinite is usually not monochromatic



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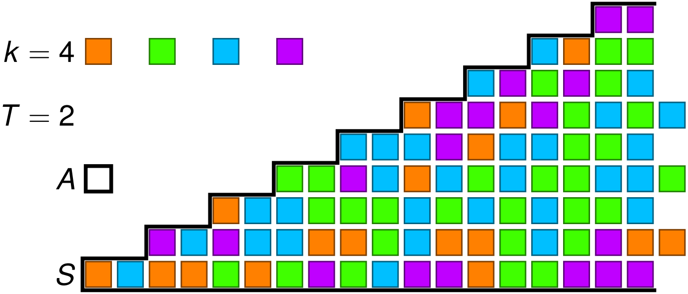

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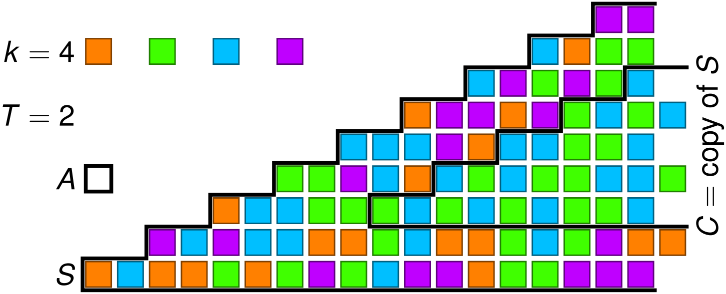

The infinite is usually not monochromatic

Ordinals:

- $T(1, \omega^\alpha) = 1$ for every ordinal α [Fraïssé]
- $T(1, \alpha) < \infty$ for every infinite ordinal α [Fraïssé]

Scattered linear orders:

- $T(1, A) = 1$ for every additively indecomposable A [Laver]
- $T(1, S) < \infty$ for every scattered S [Laver]

Non-scattered linear orders:

- $\mathbb{Q} \not\to (\mathbb{Q})^2_2$ [Galvin]
- $T(n, \mathbb{Q}) < \infty$ for every $n \in \mathbb{N}$ [Galvin, Laver, Devlin]

Countable linear orders

α . . . a countable ordinal

S . . . a countable linear order

$\text{Spec}(S) = (T(1, S), T(2, S), T(3, S), \ldots)$

Theorem [M, Šobot]

$\text{Spec}(\alpha)$ is finite if and only if $\alpha < \omega^\omega$.

Theorem [Galvin, Laver, Devlin]

For every non-scattered S: $\text{Spec}(S)$ is always finite.

Theorem [Dasilva Barbosa, M, Nenadov]

For scattered S: $\text{Spec}(S)$ is finite if and only if $\text{rk}_{\text{Hausd}}(S) < \infty$.

Monomorphic structures

Definition [Fraïssé]

A structure S is monomorphic if all finite substructures of S of the same size are isomorphic.

Examples.

- linear orders
- Hausdorff topological spaces

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Theorem [Fraïssé; Pouzet]

A ctble rel struct $\mathcal{M} = (M, \ldots)$ is monomorphic if and only if it is quantifier-free defble in some linear order $(M, <)$.

We then say that $(M, <)$ chains $\mathcal{M}$.

Monomorphic structures

$\mathcal{M}$ ... countable monomorphic structure

$T(n, \mathcal{M})$ ... the big Ramsey degree of the unique
n-element substructure of $\mathcal{M}$

$\text{Spec}(\mathcal{M}) = (T(1, \mathcal{M}), T(2, \mathcal{M}), T(3, \mathcal{M}), \dots)$

Theorem [M, Toljić]

$\text{Spec}(\mathcal{M})$ is finite if and only if $\text{Spec}(M, \prec)$ is finite for some (and thus every) minimal linear order $\prec$ that chains $\mathcal{M}$.

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$\Uparrow$

Theorem [Fraïssé's Conjecture; Laver]

The class of all countable linear orders is a WQO under embeddability.

Monomorphic decomposition

$\mathcal{S} = (S, \ldots)$ ... a relational structure

$\{E_\alpha : \alpha < \kappa\}$ ... a partition of S

Definition [Pouzet, Thiéry]

$\{E_\alpha : \alpha < \kappa\}$ is a monomorphic decomposition of $\mathcal{S}$ if for all finite $\mathcal{A}, \mathcal{B} \leqslant \mathcal{S}$ of the same size:

$$\mathcal{A} \cong \mathcal{B} \text{ iff } |A \cap E_\alpha| = |B \cap E_\alpha| \text{ for all } \alpha < \kappa.$$

Theorem [Pouzet, Thiéry]

Every relational structure has a coarsest monomorphic decomposition which we refer to as minimal.

Structures with finite monomorphic decompositions

$\mathcal{S}$ … a countable relational structure

$\{E_1, \ldots, E_m\}$ … a finite monomorphic decomposition of $\mathcal{S}$

$\mathcal{S}[E]$ … the substructure of $\mathcal{S}$ induced by $E \subseteq S$

Theorem [M, Toljić]

$\mathcal{S}$ has finite big Ramsey degrees if and only if each $\mathcal{S}[E_i]$ does, $1 \leqslant i \leqslant m$.

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A product Ramsey statement for linear orders

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$L_1, \ldots, L_m \ldots$ countable linear orders with finite big Ramsey spectra

Theorem [M, Toljić]

For every choice of $n_1, \ldots, n_m \in \mathbb{N}$ there is a $t \in \mathbb{N}$ such that for every finite coloring

$$\chi : \mathrm{Emb}(n_1, L_1) \times \ldots \times \mathrm{Emb}(n_m, L_m) \to k$$

one can find copies $C_i \leqslant L_i$, $C_i \cong L_i$, $1 \leqslant i \leqslant m$, such that

$$\bigl|\chi\bigl(\mathrm{Emb}(n_1, C_1) \times \ldots \times \mathrm{Emb}(n_m, C_m)\bigr)\bigr| \leqslant t.$$

In other words, $T\bigl((n_1, \ldots, n_m), (L_1, \ldots, L_m)\bigr) < \infty$.

Big Ramsey degrees for structures

Some Fraïssé limits with finite big Ramsey degrees:

- $\mathbb{Q}$ [Galvin, Laver, Devlin]
- The Rado graph [Sauer]
- The Henson graphs $\mathcal{H}_n$ [Dobrinen $\to$ SE⊨OP 2018]
- The generic permutation $(\mathbb{Q}, <, \sqsubset)$ [Cameron]
- The generic partial order [Hubička]

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  $\Rightarrow$ non-forsing proof for $\mathcal{H}_3$;
  $\Rightarrow$ proofs for $\mathbb{Q}$ and the Rado graph without Milliken's strong subtree theorem

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An alternative proof for the generic partial order:

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2. $T((n, n), (\mathbb{Q}, \mathbb{Q})) < \infty$ by the Product thm for lin orders
3. The generic permutation $(\mathbb{Q}, <, \sqsubset)$ has finite big Ramsey degrees
4. The generic poset is quantifier-free definable in the generic permutation: $x \preccurlyeq y$ iff $x = y \vee (x < y \wedge x \sqsubset y)$

What next?

Dual big Ramsey degrees:

- countable limit ordinals $\rightarrow$ [Kawach, Todorčević] (modulo a restriction)
- graph-like structures $\rightarrow$ [Džuklevski, M]

What happens in case of:

1. countable ordinals in general?
2. scattered linear orders?
3. non-scattered linear orders and $\mathbb{Q}$ in particular?