Cofinal types of topological groups

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A **Tukey reduction** from a directed set $(D, \leq_D)$ to a directed set $(E, \leq_E)$ is a function $f : D \to E$ mapping **unbounded** subsets of D to **unbounded** subsets of E.

We write $D \leq_T E$ when D is **Tukey reducible** to E.

We write $D \equiv_T E$ and say that D and E are **Tukey equivalent** whenever $D \leq_T E$ and $E \leq_T D$.

Proposition

$D \leq_T E$ iff there is a function $g : E \to D$ mapping **cofinal** subsets of E to **cofinal** subsets of D.

Teorema

(Tukey, 1940) Let D and E be a directed sets. $D \equiv_T E$ iff D and E are **isomorphic** to cofinal subsets of a single directed set.

Problem

(Tukey, 1940) What are possible **cofinal types** *of directed sets?*

Proposition

(Tukey, 1940) 1 and ω are the only cofinal types of **countable** directed sets.

For a group G, the inverse mapping $In : G \to G$ is defined by the rule $In(x) = x^{-1}$, for each $x \in G$.

Definicija

A topological group G is a group with topology such that multiplication mapping $G \times G \to G$ is continuous, when $G \times G$ si given the product topology, and inverse mapping $In : G \to G$ is also continuous. An easy verification shows that G is a topological group iff the mapping $(x, y) \to xy^{-1}$ of $G \times G$ to G is continuous.

Teorema

(Birkhoff-Kakutani) Topological group G is metrizable iff it is first-countable.

We can conclude that topological group G is metrizable iff $G \leq_T \omega$.

Tukey order in the class of topological groups

Lemma

Let G be a topological group with the identity e. Let $(\mathcal{N}_1, \supseteq)$ and $(\mathcal{N}_2, \supseteq)$ be local basis of e in G. Then $\mathcal{N}_1 \equiv_T \mathcal{N}_2$.

Definition

Let G be a topological group with the identity e and D a directed set. We say that $G \leq_T D$ if there is a local base of e in G, say $(\mathcal{N}, \supseteq)$ such that $\mathcal{N} \leq_T D$.

Definition

Let G be a topological group with the identity e_G and H a topological group with the identity e_H. We say that $G \leq_T H$ if there is a local base of e_G in G, say $(\mathcal{N}_G, \supseteq)$, and a local base of e_H in H, say $(\mathcal{N}_H, \supseteq)$ such that $\mathcal{N}_G \leq_T \mathcal{N}_H$.

Tukey order in the class of topological groups

Lemma

Let G be a topological group. Then:

a) *If H is subgroup of G, then $H \leq_T G$;*

b) *If H is an open subgroup of G, then $G \equiv_T H$;*

c) *If H is a topological group and $\varphi : G \to H$ an open continuous homomorphism, then $H \leq_T G$.*

Definition

Let $\{X_i : i \in I\}$ be a collection of topological spaces and let κ be a regular infinite cardinal. We define the *κ-box topology* on $\prod_{i\in I} X_i$ as the topology given by the base

$$\left\{ \bigcap_{i\in K} \pi_i^{-1}[U_i] : K \in [I]^{<\kappa}, (\forall i \in K)\ U_i \text{ is open in } X_i \right\}.$$

Theorem

Let $\{G_i : i \in I\}$ be a collection of topological groups and κ an infinite regular cardinal. For each $i \in I$, let $\mathcal{N}_i$ be a local base of the identity e_i of the group G_i. Then the group $G = \prod_{i\in I} G_i$ with the κ-box topology is a topological group. The local base of the identity of this group is

$$\mathcal{N} = \left\{ \bigcap_{i\in K} \pi_i^{-1}[B_i] : K \in [I]^{<\kappa}, (\forall i \in K)\ B_i \in \mathcal{N}_i \right\}.$$

Tukey order in the class of topological groups

Example

Let G and H be two topological groups such that $G \not\leq_T H$ and $H \not\leq_T G$. Let H' be the topological group with the same underlying group as H, but with the discrete topology. Let $\phi : G \times H' \to H$ be the natural projection to the second coordinate. Then $G \times H' \equiv_T G$ and ϕ is a continuous homomorphism but $G \times H' \not\leq_T H$ and $H \not\leq_T G \times H'$.

Example

Continuous homomorphism doesn't preserve cofinal types of topological groups.
Let S_{ω_1} be symmetric group on ω_1. Let $\mathcal{O}_T$ be the product topology on S_{ω_1} and let $\mathcal{O}_{\omega_1}$ be the ω_1-box topology. Then $(S_{\omega_1}, \mathcal{O}_T) \equiv_T [\omega_1]^{<\omega}$ and $(S_{\omega_1}, \mathcal{O}_{\omega_1}) \equiv_T \omega_1$ and clearly $\omega_1 <_T [\omega_1]^{<\omega}$.
On the other hand $id_{S_{\omega_1}} : (S_{\omega_1}, \mathcal{O}_{\omega_1}) \to (S_{\omega_1}, \mathcal{O}_T)$ is a continuous homomorphism.

Tightness and similar properties

Definition

A topological space X is called *Fréchet space* if for every $A \subseteq X$ and $x \in \overline{A}$ there is a sequence $\{x_n : n < \omega\} \subseteq A$ converging to x.

Definition

A topological space X is *countably tight* if for every $A \subseteq X$ and every $x \in \overline{A}$ there is a countable set $C \subseteq A$ such that $x \in \overline{C}$.

Example

Topological group $G = \{x \in 2^{\omega_1} : |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| < \omega\}$, with the product topology and operation coordinatewise addition modulo 2 is Fréchet and $G \equiv_T [\omega_1]^{<\omega}$.
On the other hand, $H = 2^{\omega_1}$ with the product topology and the same operation is not countably tight and $H \equiv_T [\omega_1]^{<\omega}$.
Thus, Tukey doesn't preserve Fréchet.

Tightness and similar properties

Definition

For a topological space X, the *tightness* of X is the minimal cardinal $\kappa \geq \omega$ with the property that for every set $A \subseteq X$ and every point $x \in \overline{A}$, there is $C \subseteq A$ such that $|C| \leq \kappa$ and $x \in \overline{C}$.
The tightness of a space X is denoted by $t(X)$.

Tightness and similar properties

Lemma

(Kuzeljević-M) Let κ, λ be regular infinite cardinals and G a topological group with $t(G) = \kappa$ and such that $G \leq_T \lambda \times \kappa^+$. Then $G \leq_T \lambda$.

Corollary

Let κ be an infinite regular cardinal and G a topological group with $t(G) = \kappa$ and such that $G \leq_T \omega \times \kappa^+$. Then G is metrizable.

Corollary

If G is a countably tight topological group and $G \leq_T \omega \times \omega_1$, then G is metrizable. In particular if a countably tight group G is Tukey reducible to ω_1 then it is discrete.

Metrizability of countably tight groups

Definition

We say that a directed set $(D, \leq)$ is *strongly basically generated* if there is a metric ρ on D such that (D, ρ) is a separable metric space and that for every sequence $\{d_n : n < \omega\} \subseteq D$ converge to some $d \in D$, there is $d^\star \in D$ such that $\rho(d, d^\star) \leq \sup\{\rho(d, d_n) : n < \omega\}$ and $d_n \leq d^\star$ for each $n < \omega$.

For a topological group G, we say that G is *strongly basically generated* if $G \leq_T D$ for some strongly basically generated directed set D.

Metrizability of countably tight groups

Theorem

(Dow-Feng) Let $P = K(M)$ *for some separable metric space* M*. If* X *is a compact space with countable tightness and has a* P*-base, then* X *is first-countable.*

Theorem

(Todorčević) If G *is a Fréchet topological group such that* $G \leq_T D$*, for some basic order* D*, then* G *is metrizable.*

Theorem

(Kuzeljević-M) Suppose that X is regular, locally countably compact, and countably tight topological space. Let $x \in X$ be such that its neighborhood filter $\mathcal{F}_x$ is Tukey reducible to some strongly basically generated directed set D. Then x has a countable local base in X.

Corollary

(Kuzeljević - M) Every countably tight, locally countable compact, and strongly basically generated topological group is metrizable.

Products

Definition

Let $\{D_i : i \in I\}$ be a collection of partially ordered sets. Suppose that D_i has a minimum 0_i for each $i \in I$. Let κ be an infinite regular cardinal. We define the κ-support product of posets D_i as follows:

$$\prod_{i \in I}^{\kappa-supp} D_i = \left\{ x \in \prod_{i \in I} D_i : |\{i \in I : x_i \neq 0_i\}| < \kappa \right\}.$$

Theorem

Let κ be an infinte regular cardinal. Let $\{G_i : i \in I\}$ be a collection of topological groups such that $G_i \equiv_T D_i$ where D_i is a directed set with the minimum 0_i, for each $i \in I$. Suppose that $G = \prod_{i \in I} G_i$ with the κ-box topology, and that $D = \prod_{i \in I}^{\kappa-supp} D_i$. Then $G \equiv_T D$.

THANK YOU!