Weakly o-minimal types

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General setting, notation, terminology

$(\mathfrak{C}, <, \dots)$: a monster model (κ-saturated and strongly κ-homogeneous for some big enough cardinal κ) of a complete first-order theory T saying that $<$ is a linear order.

$S_x(A)$: the space of types, i.e. maximal consistent sets of formulae, in variables x with parameters from $A \subseteq \mathfrak{C}$.

$S_p(B)$: the space of extensions of a type $p \in S_x(A)$ in $S_x(B)$, where $B \supseteq A$.

$\varphi(\mathfrak{C})$: the set of solutions of a formula $\phi(x)$ in $\mathfrak{C}^{|x|}$.

$p(\mathfrak{C})$: the set of realizations of a type $p(x)$ in $\mathfrak{C}^{|x|}$.

$X \subseteq p(\mathfrak{C})$ is *relatively definable* if $X = \varphi(\mathfrak{C}) \cap p(\mathfrak{C})$.

o-minimality and generalizations

Definition

- (a) T is *o-minimal* if every $\mathfrak{C}$-definable subset of $\mathfrak{C}$ is a finite union of intervals (sets (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ where $a \leqslant b \in \mathfrak{C} \cup \{\pm\infty\}$).
- (b) T is *weakly o-minimal* if every $\mathfrak{C}$-definable subset of $\mathfrak{C}$ is a finite union of convex sets.
- (c) T is *[weakly] quasi-o-minimal* if every $\mathfrak{C}$-definable subset of $\mathfrak{C}$ is a Boolean combination of $\emptyset$-definable subsets and intervals [convex sets].

	$\Rightarrow$	quasi-o-minimal	$\Rightarrow$	
o-minimal				weakly quasi-o-minimal
	$\Rightarrow$	weakly o-minimal	$\Rightarrow$	

Example

Weakly quasi-o-minimal theories include all (coloured) linear orders.

Monotonicity Theorems

Theorem (Pillay, Steinhorn)

If $Th(\mathfrak{C}, <, \dots)$ *is o-minimal, and* $f : \mathfrak{C} \to \mathfrak{C}$ *is a definable function, then there is a finite definable convex partition of* $\mathfrak{C}$ *such that* f *is either constant or strictly monotone on each member of the partition.*

Theorem (Macpherson, Marker, Steinhorn)

If $Th(\mathfrak{C}, <, \dots)$ *is weakly o-minimal, and* $f : \mathfrak{C} \to \mathfrak{C}$ *is a definable function, then there is a finite definable convex partition of* $\mathfrak{C}$ *such that* f *is either locally constant or locally strictly monotone on each member of the partition.*

Weakly o-minimal types and pairs

Definition

- (a) A type $p \in S(A)$ is *weakly o-minimal* if there exists a relatively A-definable linear order $<$ on $p(\mathfrak{C})$ such that every relatively $\mathfrak{C}$-definable subset of $p(\mathfrak{C})$ is a finite union of $<$-convex subsets of $p(\mathfrak{C})$.
- (b) If a relatively A-definable linear order $<$ witnesses that p is weakly o-minimal, we also say that $(p, <)$ is a *weakly o-minimal pair over A*.

Example

If $Th(\mathfrak{C}, <, \dots)$ is weakly quasi-o-minimal, then $(p, <)$ is a weakly o-minimal pair for every $p \in S_1(A)$.

A characterization

Lemma

If $(p, <)$ is a weakly o-minimal pair over A, $B \supseteq A$ and $q \in S_p(B)$, then $(q, <)$ is a weakly o-minimal pair over B. In particular, extensions of weakly o-minimal types are weakly o-minimal.

Lemma

1. *If $(p, <)$ is a weakly o-minimal pair over A, $B \supseteq A$ and $q \in S_p(B)$, then $q(\mathfrak{C})$ is a $<$-convex subset of $p(\mathfrak{C})$.*
2. *If $p \in S(A)$ and $<$ is a relatively A-definable linear order on $p(\mathfrak{C})$ such that for every $B \supseteq A$ and every $q \in S_p(B)$, $q(\mathfrak{C})$ is a $<$-convex subset of $p(\mathfrak{C})$, then $(p, <)$ is a weakly o-minimal pair.*

On equivalences

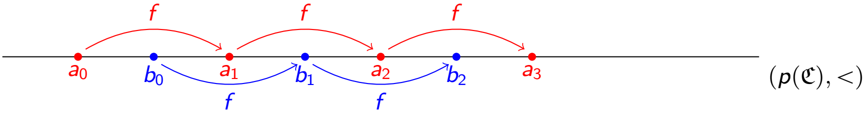
For $p \in S(A)$ let $\mathcal{E}_p$ be the set of all relatively A-definable equivalences on $p(\mathfrak{C})$.

Lemma

If $(p, <)$ is a weakly o-minimal pair over A, then:

1. *each $E \in \mathcal{E}_p$ is $<$-convex;*
2. *$(\mathcal{E}_p, \subseteq)$ is a linear order.*

Proof of (i). Suppose that $a_0, b_0, a_1 \models p$ are such that $a_0 < b_0 < a_1$, $E(a_0, a_1)$ and $\neg E(a_0, b_0)$. Take $f \in Aut(\mathfrak{C}/A)$ such that $f(a_0) = a_1$; set $a_{n+1} := f(a_n)$ and $b_{n+1} = f(b_n)$. Then $a_n < b_n < a_{n+1}$, $E(a_0, a_n)$ and $\neg E(a_0, b_n)$.

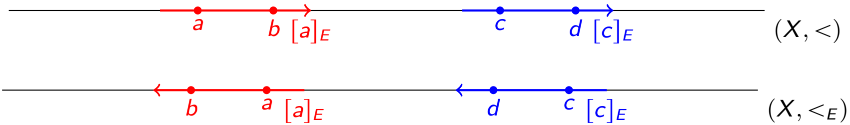


So, $[a_0]_E$ has infinitely many $<$-convex parts; a contradiction. □

Defining new orders

Let $(X, <)$ be a linear order and E a $<$-convex equivalence relation on X. We define a new linear order $<_E$ on X by:

$$a <_E b \;:\Leftrightarrow\; (E(a,b) \wedge b < a) \vee (\neg E(a,b) \wedge a < b).$$



If $\vec{E} = (E_1, E_2, \ldots, E_n)$ is a sequence of pairwise $\subseteq$-comparable $<$-convex equivalence relations on X, the previous construction can be iterated, and we define $<_{\vec{E}}$.

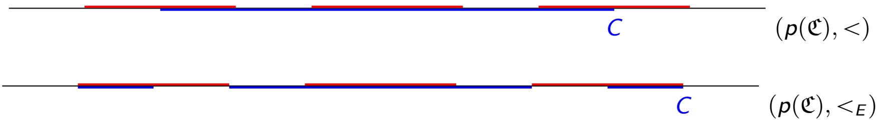
Remark. If $(p, <)$ is a weakly o-minimal pair over A, and $\vec{E} \in \mathcal{E}_p^n$, then $<_{\vec{E}}$ is a relatively A-definable linear order on $p(\mathfrak{C})$.

The main technical result

Lemma

If $(p, <)$ is a weakly o-minimal pair over A, and $\vec{E} \in \mathcal{E}_p^n$ then $(p, <_{\vec{E}})$ is a weakly o-minimal pair over A too.

Proof.



Theorem

If $(p, <)$ is a weakly o-minimal pair over A and $\lhd$ a relatively A-definable linear order on $p(\mathfrak{C})$, then $\lhd = <_{\vec{E}}$ for some increasing $\vec{E} \in \mathcal{E}_n$.

Therefore, for a type $p \in S(A)$, being weakly o-minimal doesn't depend on the choice of relatively A-definable linear order on $p(\mathfrak{C})$.

A corollary

Theorem

$Th(\mathfrak{C}, <, \dots)$ is weakly quasi-o-minimal iff every $p \in S_1(\emptyset)$ is weakly o-minimal.

Furthermore, if $\lhd$ is an $\emptyset$-definable linear order on $\mathfrak{C}$, then $Th(\mathfrak{C}, \lhd, \dots)$ is weakly quasi-o-minimal.

Monotonicity theorems for weakly o-minimal types

Theorem (Weak monotonicity)

Let $(p, <)$ be a weakly o-minimal pair over A, $(D, \lhd)$ an A-definable linear order, and $f : p(\mathfrak{C}) \to D$ a relatively A-definable nonconstant function.

1. *There is $\vec{E} \in \mathcal{E}_p^n$ such that f is $(<_{\vec{E}}, \lhd)$-increasing (meaning that $a <_{\vec{E}} b$ implies $f(a) \trianglelefteq f(b)$).*
2. *There is an increasing sequence $\vec{F}$ of $\lhd$-convex A-definable equivalences on D such that f is $(<, \lhd_{\vec{F}})$-increasing.*

Monotonicity theorems for weakly o-minimal types

Under the same assumptions:

Theorem (Local monotonicity)

There is a non-trivial equivalence $E \in \mathcal{E}_p$ such that f is either constant of strictly $(<, \lhd)$-monotone on each E-class.

Theorem (Upper monotonicity)

There is a convex relatively A-definable equivalence E on $p(\mathfrak{C})$, such that $E \neq p(\mathfrak{C})^2$ and one of the following two conditions holds for all x_1, x_2 realizing p:

$$[x_1]_E < [x_2]_E \Rightarrow f(x_1) \lhd f(x_2) \quad \text{or} \quad [x_1]_E < [x_2]_E \Rightarrow f(x_1) \rhd f(x_2).$$

In weakly o-minimal theories

Theorem

Suppose that $Th(\mathfrak{C}, <, \dots)$ is weakly o-minimal, $(D, \lhd)$ is an A-definable linear order and $f : \mathfrak{C} \to D$ is an A-definable function. Then:

1. *There exists a finite convex A-definable partition of $\mathfrak{C}$ and an increasing sequence of A-definable convex equivalence relations $\vec{E}$ on $\mathfrak{C}$ such that f is $(<_{\vec{E}}, \lhd)$-increasing on each member of the partition.*
2. *There exists a finite convex A-definable partition of $\mathfrak{C}$ and a convex A-definable equivalence relation E on $\mathfrak{C}$ with finitely many finite classes, such that f is constant or strictly $(<, \lhd)$-monotone on each class.*

Thank you!