Polish ultrametric spaces, their isometry groups, and generalized wreath products

Luca Motto Ros

Department of Mathematics "G. Peano"
University of Turin, Italy
luca.mottoros@unito.it
https://sites.google.com/site/lucamottoros/

Joint work with R. Camerlo and A. Marcone

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Remark. A related problem posed by Pestov, asking for a characterization of all subgroups of isometry groups of ultrametric spaces, was solved by Lemin and Smirnov in 1986.

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3. Suppose that X is a W-space, i.e. a Polish ultrametric space satisfying the following two conditions:
   - X is locally non-rigid;
   - the homogenous classes of X have **exact distances**, i.e. for any two such classes $[x]$ and $[y]$ there are $x' \in [x]$ and $y' \in [y]$ such that $d(x', y') = \mathrm{dist}([x], [y])$.

   Then $\mathrm{Iso}(X)$ can be described using a natural variant of Holland's generalized wreath product (Malicki, 2014).

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4. Proper (= closed balls are compact) spaces $\rightsquigarrow$ all locally compact Polish groups (Gao-Kechris, 2003 + Malicki-Solecki, 2009)

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1. $\mathrm{Iso}(X)$ is isomorphic to a closed subgroup of $\mathrm{Sym}(\omega)$ (folklore). Indeed, $\mathrm{Iso}(X) \cong \mathrm{Aut}(T)$ for some R-tree T; the converse holds as well (Gao-Shao, 2011).

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4. If X is Heine-Borel, then $\mathrm{Iso}(X)$ is the closure of an increasing union of compact subgroups, and hence it is amenable (Gao-Kechris, 2003).

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4. if $t, t' \in T$ are $\leq_T$-incomparable, then the set $\{\ell \geq_L \mathrm{lev}_T(t), \mathrm{lev}_T(t') \mid t|_\ell \neq t'|_\ell\}$ has a maximum in L, denoted by $\mathrm{split}(t, t')$.

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Given an L-tree T, let

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$$b(\ell) \leq_T b(\ell') \text{ for every } \ell \leq_L \ell'\}$$

be the **body** of T, and call its elements **branches** of T. We say that T is **pruned** if for every $t \in T$ there is $b \in [T]$ such that $b(\mathrm{lev}_T(t)) = t$.

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An **embedding** between two L-trees $(T, \leq_T)$ and $(S, \leq_S)$ is an injection $f \colon T \to S$ such that $\mathrm{lev}_S(f(t)) = \mathrm{lev}_T(t)$ and $t \leq_T t' \iff f(t) \leq_S f(t')$, for all $t, t' \in T$. An **isomorphism** is a surjective embedding, and it is called **automorphism** when $T = S$. The group of automorphisms of T is denoted by $\mathrm{Aut}(T)$.

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4. $G \cong \mathrm{Aut}(T)$ for some countable pruned L-tree T.

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- If Δ is an antichain and $\mathcal{S} = \mathcal{P}(\Delta)$, then $\mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta = \prod_{\delta \in \Delta} H_\delta$.

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1. if $x|_\delta = y|_\delta$, then $g(x)|_\delta = g(y)|_\delta$;
2. the map $i \mapsto g(x_i^\delta)(\delta)$ is a permutation of N_δ belonging to H_δ.

Examples:

- If $\Delta = 2$ and $\mathcal{S} = \mathcal{P}(\Delta)$, then $\mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta = H_0 \mathrm{\ Wr\ } H_1$.
- If Δ is an antichain and $\mathcal{S} = \mathcal{P}(\Delta)$, then $\mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta = \prod_{\delta \in \Delta} H_\delta$.
- If Δ is an antichain and $\mathcal{S} = \mathrm{Fin}$, then $\mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta = \bigoplus_{\delta \in \Delta} H_\delta$.

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We add a fourth option: Δ arbitrary and $\mathcal{S} = \mathrm{LF}$, where

$$A \in \mathrm{LF} \iff A \cap \{\gamma \in \Delta \mid \gamma \geq \delta\} \in \mathrm{Fin} \text{ for all } \delta \in \Delta.$$

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Clearly

$$\mathrm{Fin} \subseteq \mathrm{LF} \subseteq \mathrm{UM} \subseteq \mathrm{Max},$$

and $\mathrm{UM} = \mathrm{LF}$ when Δ is (the underlying order of) an L-tree.

Generalized wreath products as topological groups

We can equip each generalized wreath product $\mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta$ with the (group) topology whose neighborhood system for the identity is generated by the sets of the form

$$U_{x,\gamma} = \left\{ g \in \mathrm{Wr}^{\mathcal{S}}_{\delta \in \Delta} H_\delta \mid g(x)|_\gamma = x|_\gamma \right\},$$

where $x \in S$ and $\gamma \in \Delta$.

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It is not difficult to verify that the topology of $\mathrm{Wr}^{\mathcal{S}}_{\delta\in\Delta} H_\delta$ is Hausdorff. If $|\Delta| \leq \aleph_0$ and $\mathcal{S} \subseteq \mathrm{LF}$, then this topology is second-countable, and hence metrizable by the Birkhoff-Kakutani theorem.

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Theorem (Camerlo-Marcone-M.)

If $|\Delta| \leq \aleph_0$ and each H_δ is a closed transitive subgroup of $\mathrm{Sym}(N_\delta)$, then

$$\mathrm{Wr}^{\mathrm{LF}}_{\delta \in \Delta} H_\delta$$

is isomorphic to a closed subgroup of $\mathrm{Sym}(\omega)$, and it is thus a Polish group.

The homogeneous case

A metric space X is **homogeneous** if for every $x, y \in X$ there is $\varphi \in \mathrm{Iso}(X)$ such that $\varphi(x) = y$.

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The proof goes as follows:

- The functor $\mathcal{F}$ from Polish ultrametric spaces to L-trees specializes to a functor preserving homogeneity. This proves ❶ $\Rightarrow$ ❷.

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- The functor $\mathcal{F}$ from Polish ultrametric spaces to L-trees specializes to a functor preserving homogeneity. This proves ❶ $\Rightarrow$ ❷.
- To each pruned L-tree T, we can associate its **condensed tree** $\Delta(T)$: this is the quotient of T with respect to the equivalence relation

$$t \sim s \iff \varphi(t) = s \text{ for some } \varphi \in \mathrm{Aut}(T),$$

  where for $\delta, \delta' \in \Delta(T)$ we set $\mathrm{lev}_{\Delta(T)}(\delta) = \mathrm{lev}(t)$ for some (equivalently, any) $t \in \delta$, and $\delta \leq_{\Delta(T)} \delta' \iff t \leq_T t'$ for some $t \in \delta$ and $t' \in \delta'$.

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where for $\delta = [t]$ we set

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- For ❸ $\Rightarrow$ ❶, there is a natural way to equip the domain S of $\mathrm{Wr}^{\mathrm{LF}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$ with a complete ultrametric d so that

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This concludes the proof.

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The latter is just Hall's generalized wreath product restricted to countable objects!

More homogeneous classes

Recall Malicki's result (2014): If X is locally non-rigid and its homogeneous classes have exact distances, then $\mathrm{Iso}(X)$ is of the form $\mathrm{Wr}^{\mathrm{UM}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$.

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In the context of L-trees, such requirement can be translated to:

($\star$) For every $\ell \in L$ and every $\bar{t}, t \in \mathrm{Lev}_\ell(T)$, if for every $\ell' >_L \ell$ there is $t' \sim t$ such that $t'|_{\ell'} = \bar{t}|_{\ell'}$, then there is $t'' \sim t$ such that $t''|_{\ell'} = \bar{t}|_{\ell'}$ for all $\ell' >_L \ell$.

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5. $G \cong \mathrm{Wr}^{\mathrm{LF}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta) = \mathrm{Wr}^{\mathrm{UM}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, for Δ the underlying order of some L-tree.

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This is optimal: it strengthens Malicki's result and provides its converse.

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A few words on the proof:

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- The equivalence among conditions 1–4 follows from the fact that the functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{U}$ specialize to functors that preserve the extra requirements ("exact distances" and condition $(\star)$, respectively).

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- The equivalence among conditions ❶–❹ follows from the fact that the functors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{U}$ specialize to functors that preserve the extra requirements ("exact distances" and condition $(\star)$, respectively).
- The implication ❹ $\Rightarrow$ ❺ is an elaboration of the argument used in the homogeneous case. In particular, $\Delta = \Delta(T)$ is again the condensed tree of T, which now is just a partial order; the requirement on "exact distances" ensures that $\Delta(T)$ is again an L-tree.

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- To conclude the proof, this time we prove 5 $\Rightarrow$ 4. This is done by "inverting" the process in the previous item, exploiting the hypothesis that Δ is already an L-tree and the fact that we use $\mathcal{S} = \mathrm{LF}$.

Back to wreath products

To cover the general case, we further generalize the wreath products appearing in the previous theorems. There are two key observations:

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  1. if $x|_\delta = y|_\delta$, then $g(x)|_\delta = g(y)|_\delta$,

  every $g \in \mathrm{Wr}^{\mathrm{LF}}_{\delta \in \Delta} H_\delta$ induces corresponding "local" maps $g_\delta \colon Y_\delta \to Y_\delta$, for every $\delta \in \Delta$.

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the maps g_δ are permutations of Y_δ which commute with the restriction operations $Y_\delta \to Y_\gamma \colon y \mapsto y|_\gamma$.

Back to wreath products

Definition

Let $\langle \Delta, N \rangle$ be a skeleton, and let $(H_\delta)_{\delta \in \Delta}$ be a family of transitive permutation groups over the corresponding sets N_δ. We denote by

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the subgroup of $\prod_{\delta \in \Delta} \mathrm{Sym}(Y_\delta)$ consisting of those $(h_\delta)_{\delta \in \Delta}$ satisfying the following conditions, for every $\delta \in \Delta$ and $y \in Y_\delta$:

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Theorem (Camerlo-Marcone-M.)

$$\mathrm{Wr}^{\mathrm{LF}}_{\delta \in \Delta} H_\delta \cong \widetilde{\mathrm{Wr}}^{\mathrm{LF}}_{\delta \in \Delta} H_\delta$$

Projective wreath products

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Definition

A **system of projections** (over a skeleton $\langle \Delta, N \rangle$) is a family $\boldsymbol{\pi} = (\pi_{\delta\gamma})_{\gamma \geq \delta}$ of surjective maps $\pi_{\delta\gamma} \colon Y_\delta \to Y_\gamma$ such that for all $y, y' \in Y_\delta$ and $\beta \geq \gamma \geq \delta$

1. $\pi_{\delta\gamma}(y) = \pi_{\delta\gamma}(y')$ if and only if $y|_\gamma = y'|_\gamma$;
2. $\pi_{\gamma\beta} \circ \pi_{\delta\gamma} = \pi_{\delta\beta}$.

(It follows that each $\pi_{\delta\delta}$ is the identity on Y_δ.)

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Let $\langle \Delta, N \rangle$ be a skeleton, and let $(H_\delta)_{\delta \in \Delta}$ be a family of transitive permutation groups over the corresponding sets N_δ. We denote by

$$\widetilde{\mathrm{Wr}}^{\mathrm{LF}}_{\delta \in \Delta} H_\delta$$

the subgroup of $\prod_{\delta \in \Delta} \mathrm{Sym}(Y_\delta)$ consisting of those $(h_\delta)_{\delta \in \Delta}$ satisfying the following conditions, for every $\delta \in \Delta$ and $y \in Y_\delta$:

1. $h_\gamma(y|_\gamma) = h_\delta(y)|_\gamma$ for every $\gamma \geq \delta$;
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5. $G \cong \widetilde{\mathrm{Wr}}^{\mathrm{LF},\boldsymbol{\pi}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, for Δ the underlying order of some L-tree and $\boldsymbol{\pi}$ some system of projections over the skeleton $\langle \Delta, N \rangle$.

Summing up...

Isometry groups	**Automorphism groups**	**Wreath products**
discrete homogeneous	homogeneous L-trees, L with a minimum	$\mathrm{Wr}^{\mathrm{Fin}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, with Δ linear
homogeneous	homogeneous L-trees	$\mathrm{Wr}^{\mathrm{LF/UM}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, with Δ linear
homog. classes with exact distances [1]	special L-trees with property $(\star)$	$\mathrm{Wr}^{\mathrm{LF/UM}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, with Δ an L-tree
all	L-trees	$\widetilde{\mathrm{Wr}}^{\mathrm{LF},\boldsymbol{\pi}}_{\delta \in \Delta} \mathrm{Sym}(N_\delta)$, with Δ an L-tree
(perfect) locally compact		
(uniformly) discrete		

[1] Here we can further add either "perfect locally compact", or "uniformly discrete".

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Thank you for your attention!