On universally Baire Sets

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What is a universally Baire sets

Definition (Fenq-Magidor-Woodin)

$A \subseteq \mathbb{R}$ is a universally Baire set if for every topological space X with a regular open base and every continuous function $f\colon X \to \omega^\omega$, the preimage $f^{-1}[A]$ has the Baire property in X.

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Definition

Suppose X and Y are sets and $T \subseteq \cup_{n\in\omega} X^n \times Y^n$. T is called a tree on $X \times Y$ if T is closed under initial segments, i.e., if $(s, t) \in T$ and $n \leq \mathrm{dom}(s)$ then $(s \restriction n, t \restriction n) \in T$.

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Notation

Suppose T is a tree on $X \times Y$.

1. $[T] \subseteq X^\omega \times Y^\omega$ is the set of branches of T, i.e., $(a, b) \in [T]$ if and only if for all $n \in \omega$, $(a \restriction n, b \restriction n) \in T$.

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1. $[T] \subseteq X^\omega \times Y^\omega$ is the set of branches of T, i.e., $(a, b) \in [T]$ if and only if for all $n \in \omega$, $(a \upharpoonright n, b \upharpoonright n) \in T$.
2. $p[T] = \{a \in X^\omega : \exists b((a, b) \in [T]\}$.

What is a universlly Baire set

Definition

Suppose κ is an uncountable cardinal and (T, S) are two trees on $\omega \times \kappa$. (T, S) are called κ-complementing trees if for all posets $\mathbb{P}$ of size $< \kappa$ and for all V-generic $g \subseteq \mathbb{P}$, in $V[g]$, $p[T] = \mathbb{R} - p[S]$.

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Theorem (Fenq-Magidor-Woodin)

A set of reals A is universally Baire if and only if for every uncounatble κ, there is a κ-complementing pair of trees (T, S) such that $A = p[T]$.

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1. Why do we study universally Baire sets?
2. How does the study of universally Baire sets connect with other set theoretic themes?

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Answer: universally Baire sets form the largest collection of sets of reals that are immune to forcing: there cannot be independence results about them.

Remark

For the above to make sense we need to be able to *interpret* universally Baire sets in generic extensions.

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Interpreting universally Baire sets in generic extensions

Definition

Suppose A is a κ-unievrsally Baire set as witnessed by (T, S) (so $A = p[T]$), $\mathbb{P}$ is a poset of size $< \kappa$ and $g \subseteq \mathbb{P}$ is V-generic. Then $A_g^{T,S} = (p[T])^{V[g]}$.

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Proposition

Suppose $A, \kappa, (T, S), \mathbb{P}$ and g are as above. Then $A_g^{T,S}$ is independent of g.

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Proposition

Suppose $A, \kappa, (T, S), \mathbb{P}$ and g are as above. Then $A_g^{T,S}$ is independent of g. More precisely, if (U, W) is another pair of κ-complementing trees such that $A = p[U]$, then

$$A_g^{T,S} = A_g^{U,W}$$

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Proposition

Suppose $A, \kappa, (T, S), \mathbb{P}$ and g are as above. Then $A_g^{T,S}$ is independent of g. More precisely, if (U, W) is another pair of κ-complementing trees such that $A = p[U]$, then

$$A_g^{T,S} = A_g^{U,W}$$

or equivalently

$$(p[T])^{V[g]} = (p[W])^{V[g]}$$

.

No independence about universally Baire sets

Notation
$A_g = A_g^{T,S}$.

Theorem (Woodin)
Assume there is a class of Woodin cardinals and A is a universally Baire set. Then for all V-generic g there is en elementary embedding

$$j : L(A, \mathbb{R}) \to L(A_g, \mathbb{R}_g)$$

such that $j(A) = A_g$, and moreover, $L(A, \mathbb{R}) \models$ "Axiom of Determinacy" (AD).

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3. (Martin-Steel-Woodin) Assume class of Woodin cardinals. Then every set of reals in $L(\mathbb{R})$ is universally Baire, and much more (e.g. every set of reals in $L(\mathbb{R}^{\#})$ and etc.).

Does $A \mapsto A_g$ preserve meaning?

Question

Suppose ϕ is a formula, A is a universally Baire set, g is V-generic and suppose $A = \{x \in \mathbb{R} : V \models \phi[x]\}$. Does it follow that

$$A_g = \{x \in \mathbb{R}_g : V[g] \models \phi[x]\}?$$

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Answer: in general no for trivial reasons but...

Generically correct formulas

Definition

Suppose ϕ is a formula and κ is an uncountable cardinal. We say ϕ is κ-generically correct if for every $\theta > 2^\kappa$, there is $A \subseteq H_\theta$ such that for a club of elementary substructures X of $(H_\theta, A, \in)$, letting $\pi_X : N_X \to X$ be the transitive collapse of X, whenever $\mathbb{P} \in N_X$, $g \in V$ and $a \in \mathbb{R} \cap N_X[g]$ are such that

1. $N_X \models |\mathbb{P}| < \kappa_X (= \pi_X^{-1}(\kappa))$ and
2. $g \subseteq \mathbb{P}$ is N_X-generic,

$$N_X \models \phi[a] \leftrightarrow V \models \phi[a]$$

.

Generically correct formulas

Theorem (folklore?)

Suppose $A \subseteq \mathbb{R}$ and κ is an uncountable cardinal. Then the following are equivalent:

1. *A is κ-universally Baire.*
2. *A is definable via a κ-generically correct formula*

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1. *A is κ-universally Baire.*
2. *A is definable via a κ-generically correct formula i.e. there is a κ-generically correct formula ϕ such that for every V-generic g,*

$$A_g = \{x \in \mathbb{R}_g : V[g] \models \phi[x]\}.$$

Example

Assume class of Woodin cardinals. Then the formula ϕ that defines $x^{\#}$ is κ-generically correct for each κ. Thus, if $A = \{x \in \mathbb{R} : \exists y \in \mathbb{R}(x = y^{\#})\}$ and g is generic then in $V[g]$, $A_g = \{x \in \mathbb{R}_g : \exists y \in \mathbb{R}_g(x = y^{\#})\}$.

Sealing

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Definition

Sealing is the conjunction of the following clauses.

1. There is a class of Woodin cardinals.
2. $L(uB, \mathbb{R}) \models \mathrm{AD}^+$.
3. For all V-generic g and $V[g]$-generic h, there is
$$j : L(uB_g, \mathbb{R}_g) \to L(uB_{g*h}, \mathbb{R}_{g*h})$$
such that for all $A \in uB_g$, $j(A) = A_h$.

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Theorem (Woodin)

Suppose κ is a supercompact cardinal and there is a class of Woodin cardinals. Let $g \subseteq Coll(\omega, 2^{2^\kappa})$ be V-generic. Then Sealing *holds in $V[g]$.*

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Remark

It is not known if Sealing *is a consequence of some large cardinal. However, if some large cardinal implies* Sealing *then it is unlikely that there will be an inner model theory for that large cardinal.*

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1. They are fun!
2. This is the largest collection of sets of reals that is immune to forcing.
3. Under large cardinals there cannot be independence results about each individual uB set.
4. After performing an initial collapse, there cannot be independence results about the collection of universally Baire sets.

Okay, but

How does the study of universally Baire sets connect with other set theoretic themes?

Universally Baire sets as a complexity hierarchy: Wadge reducibility

Notation

For sets of reals A, B, $A \leq_W B$ if A is a continuous preimage of B.

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Theorem (Wadge)

Assume $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ are two sets of reals such that $L(A, B, \mathbb{R}) \vDash AD$. Then either $A \leq_W B$ or $B \leq_W \mathbb{R} - A$.

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Theorem (Martin)

Assume AD. Then $\leq_W$ is a well-founded relation.

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Assume there is a class of Woodin cardinals and A is a universally Baire set. Then the Wadge rank of A, $w(A)$, is the (ordinal) rank of A in $\leq_W$.

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Assume there is a class of Woodin cardinals and A is a universally Baire set. Then the Wadge rank of A, $w(A)$, is the (ordinal) rank of A in $\leq_W$. For an ordinal α, uB_α is the collection of all universally Baire set of reals A such that $w(A) = \alpha$.

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Remark

It follows from the theorem of Woodin mentioned above and the two previous theorems that for every universally Baire set A, $w(A)$ is defined.

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Universally Baire sets and inner model theory

Remark

The goal of inner model theory is to analyze uB_α with the aim of constructing concrete members of uB_α for each α.

Forcing over $L(uB)$ and its initial segments

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Theorem (Woodin, the MM model)

Assume $V = L(\mathcal{P}(\mathbb{R}))$+$\Theta_{reg}$. Let $G \subseteq \mathbb{P}_{max} * Add(1, \omega_3)$ be V-generic. Then $V[G] \models$ Martin's Maximum for posets of size $\mathfrak{c}$.

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Theorem (Woodin, the CH model)

Assume $V = L(\mathcal{P}(\mathbb{R}))+\Theta_{reg}$. Let $G \subseteq Coll(\omega_1, \mathbb{R}) * Add(1, \omega_2)$ be V-generic. Then $V[G] \models$ CH+"There is an ω_1-dense ideal on ω_1".

Θ_{reg}

Theorem

If there is a transitive model $M \vDash \Theta_{reg}$ such that $\mathbb{R} \subseteq M$ and Ord $\subseteq M$ then for some $\Gamma \subseteq uB$, $L(\Gamma, \mathbb{R}) \vDash \Theta_{reg}$.

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Theorem

Assume there is a Woodin cardinal that is a limit of Woodin cardinals. Then there is a transitive model $M \models \Theta_{reg}$ such that $\mathbb{R} \subseteq M$ and $Ord \subseteq M$.

Nairian Models

Definition

Assume *AD* and suppose $\lambda \leq \Theta$. Then the Nairian Model at λ, N_λ, is the model HOD_{λ^ω}, i.e., the universe consisting of sets that are hereditarily ordinal definable from a member of λ^ω.

The MM models obtained from Nairian Models

Theorem (Larson-Blue-Sargsyan)

Assume AD and suppose there is $\lambda \leq \Theta$ such that for some $n \in [1, \omega)$ the following holds:

1. $N_\lambda \vDash \lambda = \Theta^{+n}$,
2. *For each* $i \in [0, n]$, $N_\lambda \vDash$ "Θ^{+i} *is a regular cardinal".*
3. Θ^{N_λ} *is a regular cardinal.*
4. *For each* $i \in [1, n]$, $(\Theta^{+i})^{N_\lambda}$ *has cofinality at least* ω_2.

Let $G \subseteq \mathbb{P}_{max} * Add(1, \omega_3) * Add(1, \omega_4) * ... * Add(1, \omega_{2+n})$ *be* N_λ*-generic. Then* $N_\lambda[G] \vDash MM(c) + \forall i \leq n(\neg\square(\omega_{2+i}) + \neg\square_{\omega_{2+i}})$.

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Let $G \subseteq \mathbb{P}_{max} * Add(1, \omega_3) * Add(1, \omega_4) * ... * Add(1, \omega_{2+n})$ be N_λ-generic. Then $N_\lambda[G] \vDash MM(c) + \forall i \leq n(\neg\Box(\omega_{2+i}) + \neg\Box_{\omega_{2+i}})$.

Theorem

Suppose there is a Woodin cardinal that is a limit of Woodin cardinals. Then in $L(uB)$, there is $\lambda < \Theta$ such that N_λ has the properties mentioned above.

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Theorem (Sargsyan-Gappo)

Assume AD and suppose there is $\lambda \leq \Theta$ *such that for some* $n \in [1, \omega)$, N_λ *is as in the previous theorem. Let* $G \subseteq \mathbb{P}_{max} * Add(1, \omega_3) * Add(1, \omega_4) * ... * Add(1, \omega_{2+n})$ *be* N_λ*-generic. Then in* $N_\lambda[G]$ *for each* $i \in [1, n)$*, the* ω*-club filter on* ω_i, μ_i*, is an ultrafilter in HOD.*

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Remark

The above theorem answers a question of Ben Neria and Hayut, who constructed a model in which all successors of regular cardinals are ω-strongly measurable.

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Remark

Discussions with Blue and Poveda: In the above model, Weak Reflection Principle holds at each ω_{2+i} for $i \in [0, n-1)$.

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Question

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2. *(After Aspero-Schindler result) Assuming MM^{++}, is $L(Ord^{\omega_1})$ contained in a countably closed homogenous generic extension of $L(Ord^{\omega})$?*

The CH models obtained from Nairian Models

Remark

1. *The WRP (i.e. Blue-Poved-S.) and ω-club filter results (Gappa-S.) carry over.*

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1. *The WRP (i.e. Blue-Poved-S.) and ω-club filter results (Gappa-S.) carry over.*
2. *One can show that the same conclusions hold in $Coll(\omega_1, \mathbb{R}) * Add(1, \omega_2) * ...Add(1, \omega_{1+n})$.*
3. *Just change ω_2 to ω_1.*

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Remark

1. *Suppose I is an ω_1-dense ideal on ω_1 and suppose $g \subseteq Coll(\omega, \omega_1)$ is V-generic.*

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2. *Then in $V[g]$, there is $j : V \to M \subseteq V[g]$ such that $crit(j) = \omega_1^V$, $j(\omega_1) = \omega_2^V$ and*

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3. *if* $U = \{A \in V : A \subseteq \omega_1^V \text{ and } \omega_1 \in j(A)\}$ *then* U *is a* V*-ultrafilter extending the dual of I.*

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Discussions with Blue and Kasum
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2. if $i : V \to M$ is the generic embedding then $crit(i) = \omega_1^V$ and $i(\omega_1^V) = \omega_3^V$.

The CH models obtained from Nairian Models

$\mathcal{P}_{\omega_1}(X) = \{\sigma \subseteq X : |\sigma| = \aleph_0\}$.

Discussions with Blue and Kasum

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3. Moreover, there is an ω_1-dense ideal J on $\mathcal{P}_{\omega_1}(\omega_1)$ such that J is the projection of I, so that if $j : V \to Q$ is the J ultrapower and $i : V \to M$ is the I-ultrapower then there is $k : Q \to M$ such that $i = k \circ j$.

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4. Question. Does k induce an interesting ideal on ω_2^V?

The $\mathrm{cf}(\Theta^{uB})$ problem

Definition

Θ^{uB} is the supremum of all ordinals α such that there is a surjection $f : \mathbb{R} \to \alpha$ with the property that the relation $x \leq_f y$ if and only if $f(x) \leq f(y)$ is *uB*.

Question

Does MM^{++} decide $\mathrm{cf}(\Theta^{uB})$?

Theorem (Woodin)

In the standard models of MM^{++}, $\mathrm{cf}(\Theta^{uB})$ is either ω_1 or ω_2.

Sealing and the $\mathrm{cf}(\Theta_{uB})$ problem

Theorem (Blue-S.-Viale)

Each of the following three theories are consistent.

1. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_1$.
2. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_2$.
3. *Sealing*$+\mathrm{cf}(\Theta_{uB}) = \omega_3$.

Remark

Whether $\Theta^{L(\mathbb{R})} > \omega_3$ *is possible is a well-known open problem.*

Sealing+$\mathrm{cf}(\Theta_{uB}) = \omega_1$ made precise

Definition

Sealing+$\mathrm{cf}(\Theta_{uB}) = \omega_1$ is the following theory:

1. Sealing and $\mathrm{cf}(\Theta_{uB}) = \omega_1$.
2. If g is a V-generic preserving ω_1 and
$$j : L(uB, \mathbb{R}) \to L(uB_g, \mathbb{R}_g)$$
is the Sealing embedding then $j[uB]$ is Wadge cofinal in uB_g.

Adding new uB sets

Remark

If in the standard model of MM^{++}, $\mathrm{cf}(\Theta^{uB})$ is ω_2 then it must be that along the iteration there is a stage W in which $\mathrm{cf}(\Theta^{uB}) = \omega_1$ and there is a semi-proper forcing which adds a new uB set that is Wadge above all of the uB sets of W.

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Question

1. *Is there a semi-proper forcing that adds a new uB set of reals.*

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Question

1. *Is there a semi-proper forcing that adds a new uB set of reals.*
2. *Is it consistent that Namba forcing is semi-proper and $\mathrm{cf}(\Theta^{uB}) = \omega_2$?*

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Question

1. *Is there a semi-proper forcing that adds a new uB set of reals.*
2. *Is it consistent that Namba forcing is semi-proper and $\mathrm{cf}(\Theta^{uB}) = \omega_2$?*
3. *Does CH imply that semi-proper posets do not add a new uB set?*

uB_2....

There is an emerging theory of uB subsets of $\mathcal{P}(\mathbb{R})$, they are called uB_2, but that is a story for another time.

Thank you!