Intermediate models and Kinna-Wagner degrees

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The IMT

Theorem (Intermediate Model Theorem, Vopěnka-Grigorieff 70's)

Let $V \subseteq M \subseteq V[G]$ be transitive models of ZFC, *$V[G]$ a forcing extension of V. Then M is a forcing extension of V.*

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Of course not every extension is a forcing extension.

Example

$L[0^{\#}]$ is not a forcing extension of L. Many class forcing extensions of L are not (set) forcing extensions.

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But remarkably:

Theorem (Vopěnka '72)

Any set of ordinals $x \in V$ is generic over HOD^V.

The IMT

What if we consider models of ZF?

Question

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No.

Example (First Cohen Model)

Let $L \subseteq M = L(\{c_n : n \in \omega\}) \subseteq L[\langle c_n : n \in \omega\rangle]$, where $\langle c_n : n \in \omega\rangle$ is a generic sequence of Cohen reals. M does not satisfy choice, so can't be a forcing extension of L.

Symmetric systems

A technique to construct and analyse particular intermediate models of ZF is *symmetric extensions*.

Definition

A *symmetric system* is a triple $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F})$ where $\mathbb{P}$ is a forcing notion, $\mathcal{G}$ is a group of automorphisms on $\mathbb{P}$ and $\mathcal{F}$ is a *normal filter* of subgroups of $\mathcal{G}$.

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Automorphisms π of $\mathbb{P}$ can be extended naturally to $\mathbb{P}$-names by letting

$$\pi(\dot{x}) = \{(\pi(p), \pi(\dot{y})) : (p, \dot{y}) \in \dot{x}\}.$$

Symmetric extensions

Definition

A $\mathbb{P}$-name $\dot{x}$ is $\mathcal{S}$-*symmetric* if

$$\{\pi \in \mathcal{G} : \pi(\dot{x}) = \dot{x}\} \in \mathcal{F}.$$

We call $\dot{x}$ an $\mathcal{S}$-*name* or write $\dot{x} \in \mathsf{HS}_{\mathcal{S}}$ if $\dot{x}$ is hereditarily $\mathcal{S}$-symmetric.

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$$V[G]_{\mathcal{S}} = \{\dot{x}^G : \dot{x} \in \mathsf{HS}_{\mathcal{S}}\}.$$

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$$V[G]_{\mathcal{S}} = \{\dot{x}^G : \dot{x} \in \mathsf{HS}_{\mathcal{S}}\}.$$

Theorem

Let G be $\mathbb{P}$-generic over V. Then $V[G]_{\mathcal{S}} \models \mathsf{ZF}$.

Symmetric extension

It turns out that less than $\mathbb{P}$-genericity is sufficient:

Definition

Let $G \subseteq \mathbb{P}$ be a filter. Then G is *$\mathcal{S}$-generic*, or *symmetrically generic* over V, if $G \cap D \neq \emptyset$, for every dense $D \in V$ so that $\{\pi \in \mathcal{G} : \pi'' D = D\} \in \mathcal{F}$.

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Theorem (Karagila, S.)

Let H be $\mathcal{S}$-generic over V. Then $V[H]_{\mathcal{S}} \models \mathrm{ZF}$. In fact, for any $p \in H$ there is a $\mathbb{P}$-generic G over V with $p \in G$ and so that $V[H]_{\mathcal{S}} = V[G]_{\mathcal{S}}$.

Symmetric extensions

Symmetric extensions are really a generalisation of forcing extensions.

Example

Let $\mathbb{P}$ be a forcing notion. Then $\mathcal{S} = (\mathbb{P}, \{\mathrm{id}\}, \{\{\mathrm{id}\}\})$ is a symmetric system. G is $\mathcal{S}$-generic iff G is $\mathbb{P}$-generic and $V[G]_{\mathcal{S}} = V[G]$.

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A stupid example.

Example

Let $\mathcal{S} = (\mathbb{C}, \mathrm{Aut}(\mathbb{C}), \{\mathrm{Aut}(\mathbb{C})\})$. Then $0^{\#}$ corresponds to an $\mathcal{S}$-generic filter, but of course $V[0^{\#}]_{\mathcal{S}} = V$.

Symmetric extensions

Symmetric systems can be iterated:

Definition (Karagila, S.)

Let $\mathcal{S}$ be a symmetric system, $\dot{\mathcal{T}}$ an $\mathcal{S}$-name for a symmetric system. Then we define $\mathcal{S} * \dot{\mathcal{T}}$ as ...

We can do finite, countable support iterations ...

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We can do finite, countable support iterations ...
There are quotients:

Theorem (Karagila, S.)

Let $\mathcal{S}_0 \lessdot \mathcal{S}_1$, there there is an $\mathcal{S}_0$-name $\dot{\mathcal{S}_1/\mathcal{S}_0}$ so that $\mathcal{S}_0 * \dot{\mathcal{S}_1/\mathcal{S}_0}$ is equivalent to $\mathcal{S}_1$.

Symmetric extensions

A particular example is $\mathcal{S} = (\mathbb{P}, \mathcal{G}, \mathcal{F}) \lessdot (\mathbb{P}, \mathcal{G}, \langle\{\mathrm{id}\}\rangle)$. Then we also write $\mathbb{P}/\mathcal{S}$ for the quotient (which is a forcing notion).

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Let G be $\mathcal{S}$-generic over V. Then $V[G]_{\mathcal{S}} = V(\mathbb{P}/\mathcal{S})$.

- $V[G]$ is a forcing extension of $V[G]_{\mathcal{S}}$ by $\mathbb{P}/\mathcal{S}$.

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- $V[G]$ is a forcing extension of $V[G]_{\mathcal{S}}$ by $\mathbb{P}/\mathcal{S}$.

- Moreover any $\mathbb{P}/\mathcal{S}$-extension of $V[G]_{\mathcal{S}_0}$ is of the form $V[H]$, where H is $\mathbb{P}$-generic over V and

$$V[G]_{\mathcal{S}} = V[H]_{\mathcal{S}}.$$

Symmetric extensions

Recall that $V(x)$ is the smallest transitive inner model containing x as an element.

Theorem (Corollary of Grigorieff's work '75)

Let $V \subseteq M$ be models of ZF. Then the following are equivalent:

1. *M is a symmetric extension of V,*
2. *$M = V(x)$ where x is an element of some forcing extension of V.*

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In particular, the First Cohen Model $L(\{c_n : n \in \omega\})$ is a symmetric extension of L.

Back to intermediate models

Question

Let $V \subseteq M \subseteq V[G]$ be models of ZF, $V[G]$ a forcing extension of V. Then M is a ~~forcing~~ symmetric extension of V?

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No, but this is far less trivial and was unknown for a long time.

Bristol models

Definition

A *Bristol model* M is an intermediate model $V \subseteq M \subseteq V[G]$ that is not of the form $V(x)$ for any set x. In other words, M is not a symmetric extension of V.

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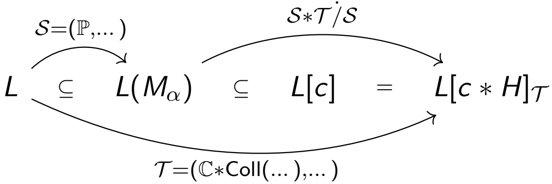
Theorem (Bristol Workshop 2011, Karagila 2017)

There is a Bristol model M, $L \subseteq M \subseteq L[c]$, where c is a Cohen real over L.

Note that each $L(M \cap V_\alpha)$ is a symmetric extension of L and that there is a proper class of such intermediate models. At the same time, there are only set many models of the form $L[c]_{\mathcal{S}}$.

Bristol models

While $L(M \cap V_\alpha)$ is of the form $L[G]_{\mathcal{S}}$ for some system $\mathcal{S}$ and a generic G, there is no reason to believe that $G = c$ or that G is even Cohen generic over L.



$\mathbb{P} \lessdot \mathbb{C} * \mathrm{Coll}(\dots)$, $\mathcal{T}$ completely forgets about the Coll part while fixing the $\mathbb{C}$ part, $\mathcal{S}$ on the other hand uses the Coll part to have particular $\mathcal{S}$-names even if they are forcing equivalent to $\mathbb{C}$-names

Kinna-Wagner Principles

What is the reason behind M not being a symmetric extension?

Definition

KWP_α says that every set injects into $\mathcal{P}^\alpha(\mathrm{Ord})$. KWP says that there is $\alpha \in \mathrm{Ord}$ so that KWP_α.

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Definition

The *Kinna-Wagner degree* of a model M is the least α such that $M \models \mathsf{KWP}_\alpha$, if it exists. Otherwise we say that M has unbounded Kinna-Wagner degree.

Kinna-Wagner principles

It turns out that for many considerations a dual notion is more useful:

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$\mathsf{KWP}_\alpha \to \mathsf{KWP}^*_\alpha \to \mathsf{KWP}_{\alpha+1}$. *For limit* α, $\mathsf{KWP}_\alpha \leftrightarrow \mathsf{KWP}^*_\alpha$.

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Theorem (Generalized Balcar-Vopěnka)

Let $M, N \models \mathsf{KWP}^*_\alpha$ *be transitive models with* $\mathcal{P}^{\alpha+1}(\mathrm{Ord})^M = \mathcal{P}^{\alpha+1}(\mathrm{Ord})^N$. *Then* $M = N$.

Kinna-Wagner principles

Theorem

The statement KWP^*_α *is invariant under forcing.*

Kinna-Wagner principles

Theorem

The statement KWP^*_α *is invariant under forcing. Both* KWP *and* $\neg$KWP *are invariant under symmetric extensions.*

In Hamkins' model logic of forcing terminology KWP^*_α is a button. These principles give a nice stratification of ZF models.

The Kinna-Wagner Conjecture

The Bristol model $L \subseteq M \subseteq L[c]$ does not satisfy KWP but L does. So M is not a symmetric extension of L.

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Conjecture (Karagila '17)

Let $V \subseteq M \subseteq V[G]$, where $V[G]$ is a forcing extension of V and $M \models \text{KWP}$. Then M is a symmetric extension of V.

Do Bristol models necessarily fail KWP?

Britol models everywhere

Recently a more general construction of Bristol models has been found:

Theorem (Hayut, Shani '24)

Let $V \models \mathrm{ZF}$ *be arbitrary and* c *a Cohen real over* V. *Then there is* $V \subseteq N \subseteq V[c]$ *so that*

1. $N \neq V(x)$ *for every* $x \in V[c]$,
2. $N \models \neg\mathrm{KWP}$,
3. AC *cannot be forced over* N *(*$M \models \neg\mathrm{SVC}$*).*

Note that 2 implies 3, but 3 implies 2 is not true in general.

Relative Kinna-Wagner degrees

Definition

Let $V \subseteq M$. Then we say that M satisfies $\mathsf{KWP}_\alpha(V)$ if for every set $x \in M$ there is an injection in M from x into some $\mathcal{P}^\alpha(V_\eta)$.

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Otherwise, we say that M has unbounded Kinna-Wagner degree over V.

Similarly we define $\mathsf{KWP}^*_\alpha(V)$.

Relative Kinna-Wagner degrees

Note that when $V \models \text{ZFC}$, then $\text{KWP}_\alpha(V)/\text{KWP}^*_\alpha(V)$ is just the same as $\text{KWP}_\alpha/\text{KWP}^*_\alpha$.

Lemma

Suppose $x \subseteq P^\alpha(V_\eta)$, for some $\eta \in \mathrm{Ord}$. Then $V(x)$ satisfies $\text{KWP}^*_\alpha(V)$.

In particular, any model of the form $V(x)$ for a set x has bounded Kinna-Wagner degree over V.

The generalized IMT

Theorem (Karagila-S. '24)

Let $V \subseteq M \subseteq V[G]$, $V[G]$ a forcing extension of V. Then the following are equivalent:

1. $M = V(x)$ *for some* $x \in M$, *i.e.* M *is a symmetric extension of* V,
2. M *has bounded Kinna-Wagner degree over* V,
3. *there is a forcing extension of* M *satisfying* $\mathsf{KWP}_0^*(V)$.

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3. *there is a forcing extension of* M *satisfying* $\mathsf{KWP}^*_0(V)$*.*

In fact, $M \models \mathsf{KWP}^*_\alpha(V)$ *iff* $M = V(x)$ *for some* $x \subseteq \mathcal{P}^\alpha(V_\eta)$*, and a specific* η *that only depends on* V*,* G *and* α*.*

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3. *there is a forcing extension of M satisfying $\mathrm{KWP}_0^*(V)$.*

In fact, $M \models \mathrm{KWP}_\alpha^*(V)$ iff $M = V(x)$ for some $x \subseteq \mathcal{P}^\alpha(V_\eta)$, and a specific η that only depends on V, G and α.

Remark

The theorem does not assume that M is definable in $V[G]$ or even amenable to $V[G]$.

The generalized IMT

Theorem (cont'd)

Also, if M satisfies $\mathsf{KWP}^*_\alpha(V)$ then $M = V(x)$, where $x \subseteq \mathcal{P}^{n\cdot\alpha}(\mathbb{P})^M$ and $n = 3$.

It is conjectured that $n = 1$ works when $\mathbb{P}$ is a cBa, but the exact computations are tedious.

The generalized IMT

In the particular case that $V \models \mathrm{ZFC}$, we obtain:

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1. *M is a symmetric extension of V,*
2. $M \models \mathrm{KWP}$,
3. $M \models \mathrm{SVC}$, *i.e. there is a forcing extension of M satifying* AC.

Genericity over HOD

Theorem (Karagila-S.)

Let $x \in V$ be arbitrary. Then x is contained in a forcing extension of HOD^V. In other words, $\mathrm{HOD}^V(x)$ is a symmetric extension of HOD^V.[1]

[1] This is not the same as $\mathrm{HOD}^V_{\{x\}}$, the sets hereditarily definable using ordinals and x as a parameter.

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The proof crucially uses the notion of a symmetrically generic filter. We find a system $\mathcal{S} \in \mathrm{HOD}^V$ and a filter G that can easily be shown to be $\mathcal{S}$-generic over HOD^V and so that x can be recovered in $\mathrm{HOD}^V[G]_{\mathcal{S}}$.

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Intermediate submodels and generic extensions in set theory.
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