More minimal non-σ-scattered linear orders

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Motivation

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The minimality question: Is it consistent that there exists a linear order $L \in \mathfrak{M}$ such that for every $K \leq L$ in $\mathfrak{M}$ we have $L \leq K$?

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The pairwise not near question: What is the largest family in $\mathfrak{M}$ of pairwise not near elements?

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This class include:

- Aronszan lines;
- Real types;
- Baumgartner types.

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- A κ-tree $(T, \lhd)$ is Aronszajn if it has no chains of size κ;
- A κ-tree $(T, \lhd)$ is Souslin if it has no chains or antichains of size κ.

From trees to linear orders

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$$s <_{\text{lex}} t \iff s \sqsubseteq t \text{ or } s(\Delta) < t(\Delta)$$

where $\Delta := \min\{\xi < \min\{\text{dom}(s), \text{dom}(t)\} \mid s(\xi) \neq t(\xi)\}$.

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If $(T, \subseteq)$ is a κ-Aronszajn tree, then $(T, <_{\text{lex}})$ is a κ-Aronszajn line.
For $(T, \subseteq)$ is a κ-Souslin tree, then $(T, <_{\text{lex}})$ contains a κ-Souslin line.

An uncountable minimal linear order

Theorem (Baumgartner 1982, D. Soukup 2019)

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Recently, Cummings, Eisworth and Moore gave a positive answer to both questions. Furthemore, they gave the first example for higher analogs of these linear orders.

Higher analog

Theorem (Cummings-Eisworth-Moore, 2023)

Consistently for each infinite cardinal λ*, there exists a minimal with respect to being non-*σ*-scattered linear order of size* λ^+*. In fact, a* λ^+*-Countryman line.*

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They used $\boxdot_\lambda$ to construct a λ^+-Aronszajn tree $(T, \subseteq)$ where $T \subseteq {}^{<\lambda^+}\omega$ which is not λ^+-Souslin such that for every antichain $X \subseteq T$ of size λ^+, $(X, <_{\text{lex}})$ is a minimal non-σ scattered linear order, i.e. for $Y \subseteq X$ such that $(Y, <_{\text{lex}})$ is a non-σ scattered, then $(X, <_{\text{lex}})$ embeds into $(Y, <_{\text{lex}})$.

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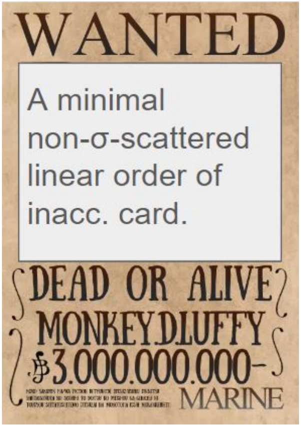
It was suggested in [CEM24] that it should be possible to extend the result to inaccessible cardinals.

Wanted


WANTED
A minimal non-σ-scattered linear order of inacc. card.
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MARINE

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Then for every antichain $X \subseteq T$ of size κ, the linear order $(X, <_{\mathrm{lex}})$ is minimal with respect to being non-σ-scattered.

Main result

Theorem

Assume κ is a regular uncountable cardinal and $P_\xi(\kappa, 2, \sqsubseteq, \kappa)$ holds for some ordinal $\xi \leq \kappa$.

Then the class $\mathfrak{M}_\kappa$ of non-σ-scattered linear orders of size κ has 2^κ-many pairwise non-near minimal elements with respect to being non-σ-scattered.

If $\xi < \kappa$, then the elements are all κ-Countryman lines.

$P_\xi(\lambda^+, 2, \sqsubseteq, \lambda^+)$ is strictly weaker than $\boxed{\diamond}_\lambda$.

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The method is known as the microscopic approach.

The pros of using the microscopic approach

- Assuming a consequence of $\diamondsuit$ which also holds in the generic extension after adding a single Cohen real to a model of CH — there exists a family of $2^{\aleph_1}$ many Countryman lines each one is minimal with respect to being non-σ-scattered and every two members of the family are not near.

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- The construction take care of the missing case, inaccessible cardinals.

- Easier to construct many trees which do not embed on a club into one another.

Antichain of linear orders

Lemma: If $(S, <_S, <_{lS})$, $(T, <_T, <_{lT})$ are two lexicographically ordered κ-Aronszajn trees, X and Y are subsets of S and T respectively, both of size κ and $\pi : (X, <_{lS}) \to (Y, <_{lT})$ is an order isomorphism, then there exists a club C such that $((X_\downarrow) \restriction C, <_S, <_{lS})$ is tree isomorphic and order isomorphic to $((Y_\downarrow) \restriction C, <_T, <_{lT})$.

- $(X_\downarrow) \restriction C = \{s \in S \mid ht_S(s) \in C \ \& \ \exists x \in X[s \leq_S x]\}$;

ϱ-modifcations

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A function $\eta : \alpha + 1 \to \mathbb{Z}$ is a modification if $\alpha < \kappa$ and η changes values only finitely many times and the changes take place at successor ordinal below α.

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Let ϱ denote the collection of all such modifications.

For $\eta \in \varrho$ and $t \in {}^{<\kappa}\omega$ with $\mathrm{dom}(\eta) \leq \mathrm{dom}(t)$ let the map $\eta * t : \mathrm{dom}(t) \to \omega$ be defined by stipulating:

$$(\eta * t)(\beta) := \begin{cases} t(\beta) + \eta(\beta), & \text{if } \beta \in \mathrm{dom}(\eta); \\ t(\beta), & \text{otherwise.} \end{cases}$$

Coherent and Uniform

Let $(T, \subseteq)$ be a tree such that $T \subseteq {}^{<\kappa}\omega$ and for all $t \in T$ and $\beta < \mathrm{ht}(t)$ we have $t \restriction \beta \in T$.

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For $\eta \in \varrho$, we define $\eta^- : \mathrm{dom}(\eta) \to \mathbb{Z}$ by letting $\eta^-(\alpha) = -\eta(\alpha)$ for $\alpha \in \mathrm{dom}(\eta)$.

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If $T \subseteq {}^{<\kappa}\omega$ is ϱ-coherent and ϱ-uniform tree, then every subset $X \subseteq T$ of size $< \kappa$ is such that $(X, <_{\mathrm{lex}})$ is σ-scattered.

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Key idea: Construct a κ-Aronszajn tree $T \subseteq {}^{<\kappa}\omega$ such that for every subtree S (downward closed and of size κ) there exists a function $\varphi : T \to S$ which is order-preserving, preserves the $<_{\text{lex}}$-order and incompatability in the tree.

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This was achieved in [CEM24] using the following:

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1. Suppose $i < \omega$ and $s, t \in T$.
   We say that *t is an i-extension of s*, written $s \subseteq_i t$, if $s \subseteq t$ and whenever $\mathrm{ht}(s) \leq \xi < \mathrm{ht}(t)$, $t(\xi) \geq i$.

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Since T is ϱ-uniform, $T_{[s,i]}$ contains a "copy" of $T_{[s,0]} = s_{\downarrow} \cup s^{\uparrow}$.

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We want a "similar" property for our constructed tree T:

- Every subtree of T contains a frozen cone.

Thank you for listening!