Weak and Strong Forms of Baumgartner's Axiom For Polish Spaces

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Every pair of countable dense sets of reals are order isomorphic. Consequently, given any pair $A, B \subseteq \mathbb{R}$ of countable, dense sets there is an order isomorphism, and hence autohomeomorphism $h : \mathbb{R} \to \mathbb{R}$ so that $h``A = B$ (" $\mathbb{R}$ is CDH").

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Denote the statement above by BA for Baumgartner's Axiom.

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Let κ be a cardinal and X a topological space. We denote by $\mathrm{BA}_\kappa(X)$ the statement that for every pair $A, B \subseteq X$ which are κ-dense there is an autohomeomorphism $h : X \to X$ so that $h``A = B$.

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Thus $\mathrm{BA} = \mathrm{BA}_{\aleph_1}(\mathbb{R})$. We are interested in general in the question of what consequences and implications between axioms like these can we expect for various κ and X?

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• For any uncountable κ it is consistent that $\mathsf{BA}_\kappa(2^\omega)$ and $\mathsf{BA}_\kappa(\omega^\omega)$ hold and in fact in ZFC both $\mathsf{BA}_\kappa(2^\omega)$ and $\mathsf{BA}_\kappa(\omega^\omega)$ hold for every $\kappa < \mathfrak{p}$. (Baldwin-Beaudoin, 1989)

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• $\mathrm{MA} + \neg\mathrm{CH}$ does not imply BA, in particular $\aleph_1 < \mathfrak{p}$ does not suffice to imply BA. (Abraham-Shelah 1981)

• For any finite $n > 1$, if X is either $\mathbb{R}^n$ or an n-dimensional compact manifold then $\mathrm{BA}_\kappa(X)$ holds for every $\kappa < \mathfrak{p}$. (Steprāns-Watson, 1989)

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• $\mathrm{MA} + \neg\mathrm{CH}$ does not imply BA, in particular $\aleph_1 < \mathfrak{p}$ does not suffice to imply BA. (Abraham-Shelah 1981)
• For any finite $n > 1$, if X is either $\mathbb{R}^n$ or an n-dimensional compact manifold then $\mathrm{BA}_\kappa(X)$ holds for every $\kappa < \mathfrak{p}$. (Steprāns-Watson, 1989)

Consequently $\mathrm{BA}_{\aleph_1}(\mathbb{R}^n)$ does not imply BA for any finite $n > 1$.

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Question

Does BA *imply* $\mathfrak{p} > \aleph_1$*?*

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Of course more generally one can ask whether BA or even $\mathrm{BA}_\kappa(X)$ implies some cardinal characteristic inequality.

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- $U(\kappa)$: there exists a universal separable metric space of size κ - which means that every other separable metric space of size κ can be homeomorphically embedded into it.

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It's not hard to see that $\mathsf{BA}_\kappa(2^\omega) \to \mathsf{BA}^-(\kappa) \to U(\kappa)$. Medini asked whether any of these arrows could be reversed and whether $U(\kappa)$ is in fact simply a theorem of ZFC (note for $\kappa = \aleph_0$ and $\kappa = 2^{\aleph_0}$ it is).

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- $\mathsf{BA}^-_\kappa(X)$: all κ dense subsets of X are homeomorphic.
- $U_{\kappa,\lambda}(X)$: there is a subset $Z \subseteq X$ so that $|Z| = \lambda$ and if $Y \subseteq X$ and $|Y| = \kappa$ then there is a continuous injection $f : Y \to Z$. "There is a set of size λ which is universal for sets of size κ".

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Obviously $\mathsf{BA}_\kappa(X) \to \mathsf{BA}^-_\kappa(X) \to U_{\kappa,\kappa}(X)$. None of these are theorems of ZFC for any $\kappa < \lambda < 2^{\aleph_0}$ and uncountable Polish space X (more on this later).

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As a consequence we get the following.

Corollary

$\mathfrak{p} > \kappa$ *implies* $\mathsf{BA}^-_\kappa(\mathbb{R})$ *and hence* $\mathsf{BA}^-_{\aleph_1}(\mathbb{R})$ *does not imply* BA.

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Uncomfortably I do not know whether $\mathsf{BA}^-_{\aleph_1}(X)$ is different from $\mathsf{BA}_{\aleph_1}(X)$ for most other uncountable Polish spaces. In particular I don't know how to separate $\mathsf{BA}_{\aleph_1}(2^\omega)$ from $\mathsf{BA}^-_{\aleph_1}(2^\omega)$.

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In fact Medini in point 1 shows that P need only be a "Cantor Crowded" separable metric space.

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Proof.

For the first bullet point, by a result of Bartoszyński and Shelah that there is a subset $X \subseteq 2^\omega$ of size $\mathfrak{b}$ which cannot be continuously mapped onto an unbounded subset of ω^ω.

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- Since every perfect Polish space is such that every basic open contains a copy of 2^ω we can find a $\mathfrak{b}$-dense copy of X as above.
- Such cannot be therefore even continuously surjected onto any unbounded set, of which we can find one also $\mathfrak{b}$-dense in any perfect Polish space (since there is a copy of ω^ω in every basic open). □

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• We can partition A into κ many disjoint, countable dense subsets, say $A = \bigcup_{\alpha\in\kappa} A_\alpha$. For each $Z \subseteq \kappa$ of size κ we get that $A_Z := \bigcup_{\alpha\in Z} A_\alpha$ is κ-dense. Thus by $\mathrm{BA}^-_\kappa(P)$ there is a homeomorphism $h_Z : A \to A_Z$.

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• By a standard result from descriptive set theory for each Z there are G_δ subsets W^0_Z and W^1_Z and a homeomorphism $\hat{h}_Z : W^0_Z \to W^1_Z$ extending h_Z.

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- By a standard result from descriptive set theory for each Z there are G_δ subsets W^0_Z and W^1_Z and a homeomorphism $\hat{h}_Z : W^0_Z \to W^1_Z$ extending h_Z. Since there are only continuum many Borel sets and continuous functions, since $2^\kappa > 2^{\aleph_0}$, there are distinct Z and Z' so that $\hat{h} := \hat{h}_Z = \hat{h}_{Z'}$.
- But this is a contradiction since $\hat{h}$ cannot homeomorphically map A onto two distinct sets. □

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- *(S.) For any regular $\kappa \leq \lambda \leq \mu$ it is consistent that $U_{\kappa,\kappa}(P)$ holds, $2^{\aleph_0} = \lambda$ and $2^\kappa = \mu$.*

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Proof of Second Point.

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• After $\aleph_2$-many steps the continuum will be $\aleph_2$, and every $\aleph_1$-sized set will appear at some initial stage. Therefore it is homeomorphic to a subset of the original ground model reals which are hence the universal set desired. Moreover $2^{\aleph_1} = \aleph_3$ by the ccc. □

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Let P be either $\mathbb{R}$, ω^ω or 2^ω. For no κ does $U_{\kappa,\kappa}(P)$ imply $\mathsf{BA}^-_\kappa(P)$. In particular in the case of $\mathbb{R}$ the axioms of BA, $\mathsf{BA}^-_{\aleph_1}(\mathbb{R})$ and $U_{\aleph_1,\aleph_1}(\mathbb{R})$ are all distinct.

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Awkwardly I do not know of any non-trivial consequences of $U_{\kappa,\lambda}(X)$ for any κ, λ or X. The following in particular seems like a nice test question which has nevertheless evaded capture.

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Does $U_{\mathfrak{d},\mathfrak{d}}(\mathbb{R})$ imply $\mathfrak{d} = 2^{\aleph_0}$?

I conjecture the answer is no.

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Proof.

Let $\{c_i \mid i \in \mu\} \subseteq 2^\omega$ be the Cohen generics over V and work in $V[c_i \mid i \in \mu]$. If there is a set $Z \subseteq 2^\omega$ of size $< \mu$ which is universal for sets of size κ then by the ccc there is a set $I \subseteq \mu$ which has size $<\mu$ and $Z \in V[c_i \mid i \in I]$. □

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If μ is uncountable and $\{c_i \mid i \in \mu\}$ are generics for adding μ many Cohen reals then in $V[c_i \mid i \in \mu]$ if $I \subseteq \mu$ is uncountable then any continuous $f : \{c_i \mid i \in I\} \to 2^\omega \cap V$ will have countable range.

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In particular co-countably many elements of $\mathcal{C}_I$ are generic over the model with W and $\hat{f}$. If $c \in \mathcal{C}_I$ is any one of these co-countably many elements then it is forced to be in the closed set $\hat{f}^{-1}(\{y\})$ for some ground model $y \in 2^\omega$ which therefore must be non-meager.

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Let P be a perfect Polish space, $X, Y \subseteq P$ with $|X| = \kappa$ for some $\kappa \leq 2^{\aleph_0}$. Say that X strongly does not embed into Y if for every $Z \subseteq X$ of size κ if $f : Z \to Y$ is continuous then the range of f has size $<\kappa$.

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We have the following.

Theorem (S.)

Suppose δ is an ordinal, P is a perfect Polish space, $X, Y \subseteq P$ with $|X| = \kappa$ and X strongly does not embed into Y. If $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \mid i < \delta \rangle$ is a finite support iteration of ccc forcing notions and for each $i < \delta$ we have that $\Vdash_i$ "$\dot{\mathbb{Q}}_i$ forces that $\check{X}$ strongly does not embed into $\check{Y}$" then $\Vdash_\delta$ "$\check{X}$ strongly does not embed into $\check{Y}$".

The Failure of $U_{\kappa,\lambda}(X)$

Using the above iteration theorem we can show the following.

Theorem

Assume GCH*. Let $\aleph_1 \leq \kappa < \mu$ be uncountable, regular cardinals. There is a ccc forcing extension in which $2^{\aleph_0} = \mu$,* $\mathsf{BA}_{\kappa'}(2^\omega)$ *holds for all $\kappa' \in [\aleph_1, \kappa]$ but $U_{\lambda,\lambda'}$ fails for all $\kappa < \lambda < \lambda' < \mu$.*

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Thus there is no "step up" from BA to U at higher cardinals by "gluing together" witnesses. The same proof works for ω^ω instead of 2^ω and, in the case of $\kappa = \aleph_1$ also for $\mathbb{R}$. These can even be all forced simultaneously in one model.

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Thus there is no "step up" from BA to U at higher cardinals by "gluing together" witnesses. The same proof works for ω^ω instead of 2^ω and, in the case of $\kappa = \aleph_1$ also for $\mathbb{R}$. These can even be all forced simultaneously in one model. The idea is that forcing instances of $\mathsf{BA}_{\aleph_1}(2^\omega)$ (say) does preserves that a set of Cohens strongly doesn't embed into the ground model. By interweaving forcing these instances with adding Cohens we get the desired model as any potential universal set is added by an initial stage and no set of Cohens added later can be forced by the tail of the forcing to embed into this candidate.

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Let's switch gears and look at stronger versions of BA.

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• $\mathrm{BA}_{\mathrm{isom}}(X)$ is the statement that for all $\aleph_1$-dense $A, B \subseteq X$ there is an isometry $f : X \to X$ so that $f``A = B$.
• $\mathrm{BA}_{\mathrm{Lip}}(X)$ is the statement that for all $\aleph_1$-dense $A, B \subseteq X$ there is a Lipschitz $f : X \to X$ with Lipschitz constant 1 so that $f``A = B$. Note we assume only that the function maps A onto B, not that it is a homeomorphism.

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Proof.

Let's do the case $X = 2^\omega$ - the other is similar. First $s \in 2^{<\omega}$. It is not hard to find continuum many $x, y \in [s]$ so that if k is least with $x(k) \neq y(k)$ then k is odd, respectively even. For each such s let O_s (respectively E_s) denote some chosen $\aleph_1$-sized subset let this.

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Let X be either 2^ω or ω^ω. $\mathrm{BA}_{Lip}(X)$ is consistent. In fact if CH holds then for any $\aleph_1$-dense $A, B \subseteq X$ there is a ccc forcing which forces the existence of a Lipschitz function $f : X \to X$ with $f``A = B$.

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The idea is similar to Baumgartner's original proof of BA. I know how to prove $\mathsf{BA}_{Lip}(X)$ consistent with large continuum - anything regular - but I don't know how to prove the consistency of the analogous statement for $\aleph_2$-dense sets.

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The axiom $\mathsf{BA}_{Lip}(X)$, for X either ω^ω or 2^ω actually proves more than (I know how to prove from) the other Baumgartner type axioms. Below let X be either ω^ω or 2^ω.

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This has an important corollary.

Theorem

$\mathrm{BA}_{Lip}(X)$ *does not follow from* $\mathfrak{p} > \aleph_1$ *and in particular does not follow from* $\mathrm{BA}_{\aleph_1}(X)$.

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I want to sketch a proof of this theorem. Recall that if $h : \omega \to \omega$ is strictly increasing then an *h-slalom* is a function $\varphi : \omega \to [\omega]^{<\omega}$ so that for all n we have $|\varphi(n)| \leq h(n)$.

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• Similarly let us write $f \in \varphi$ if for every $n < \omega$ we have $f(n) \in \varphi(n)$.

Finally for a set $A \subseteq \omega^\omega$ we say an h-slalom φ, captures A if it eventually captures every element.

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Note the point is that the cardinal doesn't depend on which h we choose - however it must be uniform for all A of size $<\kappa$.

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Using this we can show that $\mathsf{BA}_{Lip}(X)$ implies $\mathrm{add}(\mathcal{N}) > \aleph_1$. We do the case of $X = \omega^\omega$ as it is simpler but conceptually almost identical.

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Assume $\mathrm{BA}_{\mathrm{Lip}}(\omega^\omega)$. We will show that every set of size $\aleph_1$ is caught in an h-slalom for $h(n) = n2^{n+1}$. Let A be an arbitrary set of set $\aleph_1$. By possibly making it bigger we can assume that A is $\aleph_1$-dense.

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• Let $B \subseteq \omega^\omega$ defined as follows. For each $s \in \omega^{<\omega}$ let $B_s \subseteq [s]$ be an $\aleph_1$-sized set of $x \supseteq s$ so that if $k > \mathrm{dom}(s)$ then $x(k) = 0$ or $x(k) = 1$. Let $B = \bigcup_{s \in \omega^{<\omega}} B_s$. In short, B is an $\aleph_1$-dense set of functions which are eventually bounded by 2.

Strong Variations

Using this we can show that $\mathsf{BA}_{Lip}(X)$ implies $\mathrm{add}(\mathcal{N}) > \aleph_1$. We do the case of $X = \omega^\omega$ as it is simpler but conceptually almost identical.

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- By assumption there is an $f : \omega^\omega \to \omega^\omega$ so that $f``B = A$ and f is Lipschitz with Lipschitz constant 1. Fix such an f. Note that if $x, y \in \omega^\omega$, $k < \omega$ and $x \upharpoonright k = y \upharpoonright k$ then $f(x) \upharpoonright k = f(y) \upharpoonright k$ by the Lipschitz property. □

Strong Variations

Proof.

Fix $s \in \omega^{<\omega}$ and let $\varphi_s : \omega \to [\omega]^{<\omega}$ be defined by $\varphi_s(n) = \{m \mid \exists x \in B_s\ f(x)(n) = m\}$. One can show that this is a 2^{n+1}-slalom.

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• Now observe that if $x \in B_s$ then for every $n < \omega$ we have $f(x)(n) \in \varphi_s(n)$ by construction. In other words, for each $s \in \omega^{<\omega}$ the forward image $f``B_s$ is caught (totally, not eventually) by φ_s. In particular there are countably many 2^{n+1}-slaloms $\{\varphi_s \mid s \in \omega^{<\omega}\}$ so that every element of A is totally caught by (at least) one of them.

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• Now enumerate $\omega^{<\omega}$ as $\{s_n \mid n < \omega\}$ and let $\varphi(n) = \bigcup_{i<n} \varphi_{s_i}(n)$. This is a $n2^{n+1}$-slalom which eventually captures every element of A, completing the proof. □

A Ridiculous Conjecture

Returning to the original $\mathrm{BA}_{\aleph_1}(X)$, let me finish with a conjecture which, despite the its ridiculousness I actually kind of believe. More seriously it belies how little we know.

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1. X has a closed, nowhere dense subset $F \subseteq X$ so that any autohomeomorphism of X restricts to one of F and hence $\mathrm{BA}_{\aleph_1}(X)$ provably fails. (E.g. $[0, 1]$, manifolds with boundary...)

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3. The first point does not hold, X is topologically 1-dimensional and $\mathrm{BA}_{\aleph_1}(X)$ is equivalent to BA. Moreover this case implies the second one.

Thank You!
Hvala!