Countable models of a theory that interprets an infinite discrete linear order

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T – complete first-order theory in a countable L. Consider the isomorphism relation of models with domain ω, $(Mod(T), \cong)$.

T is Borel complete means that "$\cong$ is as complicated as possible".

Question 1

If T_ϕ is Borel complete for some L-formula ϕ, must T be Borel complete?

Question 2

If $T_{\bar{c}}$ is Borel complete, must T be Borel complete?

1. Usually, $T_{\bar{c}}$ is obtained "by naming parameters from a model": for $\bar{a} \in M \models T$, add $\bar{c}$ to the language and put
$$T_{\bar{c}} = T \cup \{\phi(\bar{x}) \mid M \models \phi(\bar{a})\}.$$
(Every $M \models T$ produces $\leqslant \aleph_0$ many models of $T_{\bar{c}}$.)
2. For an L-formula $\phi(x)$, T_ϕ is obtained as follows:
    1. Let $L^* = \{R_\psi \mid \psi(\bar{x})$ L-formula, R_ψ is $|\bar{x}|$-ary relation$\}$;
    2. R_ψ is interpreted in $M \models T$ as $\{\bar{a} \in M^n \mid M \models \psi(\bar{a})\}$
    3. $T^* = Th_{L*}(M)$ is constant for $M \models T$;
    4. T_ϕ – the L^*-theory of the substructure on $\phi(M)$ (constant for $M \models T$).

    (Omitting Types Theorem: Every $M_\phi \models T_\phi$ is $\phi(M)$ for some $M \models T$.)

[T] "Vaught's conjecture for theories of discretely ordered structures". arXiv:2212.13605

– T-countable, complete first-order theory;

– $I(T, \aleph_0)$ = the number of countable models of T (always $\leqslant 2^{\aleph_0}$);

Vaught's conjecture (1959)

$\aleph_0 \leqslant I(T, \aleph_0) \leqslant 2^{\aleph_0}$ is impossible, regardless of the CH.

VC holds for:

- Strongly minimal T (Marsh 1966)
- Uncountably categorical T (Morley 1967)
- Theories of colored orders (Rubin 1974)
- Theories of one unary operation (Miller 1981)
- Stable theories with Skolem functions (Lascar 1981)
- $\aleph_0$-stable T (Shelah 1984)
- o-minimal T (Mayer 1988)
- Weakly minimal T (Saffe, Buechler, Newelski 1990)
- Varieties (Hart, Starchenko, Valeriote 1994)
- Superstable of finite U-rank T (Buechler 2008)
- Binary, weakly quasi-o-minimal T (Moconja, T. 2020)
- Weakly o-minimal of finite convexity rank (Kulpeshov 2020)

– For model theorists, $I(T,\aleph_0) = 2^{\aleph_0}$ means "$\cong$ is complicated"; In general, this is weaker than Borel completeness.

VC from the point of view of first-order model theory:

The classification problem

Assuming that $I(T,\aleph_0) < 2^{\aleph_0}$, find a "reasonable" system of invariants that describes countable models up to $\cong$.

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Strong VC

Every consistent $L_{\omega_1,\omega}$-sentence has either at most countably many or perfectly many countable models.

This was formulated and proved for trees by Steel in 1976.

Theories with discrete orders

- $I(Th(\mathbb{Z},<),\aleph_0) = 2^{\aleph_0}$; models are of the form $\mathbb{L} \times \mathbb{Z}$ ($\mathbb{L}$-any linear order); $Th(\mathbb{Z},<)$ is Borel complete;
- $Th(D,<)$ Borel complete $((D,<)$ – discrete order);

Theorem [T]

If T has a definable (or interpretable) infinite discrete linear order, then $I(T,\aleph_0) = 2^{\aleph_0}$.

In the proof, we start with an infinite discrete order (defined by $\phi(x)$ and $x_1 < x_2$).

1. Without loss, assume that T is small ($|\bigcup_n S_n(T)| = \aleph_0$)
2. Name an appropriate tuple of parameters $\bar{c}$ and shrink $\phi(x)$ by an adequate $L_{\bar{c}}$-formula such that:
   – the relativization of $T_{\bar{c}}$ at ϕ, $(T_{\bar{c}})_\phi$, is interdefinable with $Th(\omega+\omega^*,<)$;
3. Then $I((T_{\bar{c}})_\phi,\aleph_0) = I(Th(\omega+\omega^*,<),\aleph_0) = 2^{\aleph_0}$ follows.

$$T \longmapsto T_{\bar{c}} \longmapsto (T_{\bar{c}})_\phi \approx Th(\omega + \omega^*, <)$$

The following is folklore for any T', ϕ, and $\bar{c}$:

- If T' is not small, then $I(T', \aleph_0) = 2^{\aleph_0}$;
- $I(T'_\phi, \aleph_0) = 2^{\aleph_0} \Rightarrow I(T', \aleph_0) = 2^{\aleph_0}$;
- $I(T'_{\bar{c}}, \aleph_0) = 2^{\aleph_0} \Rightarrow I(T', \aleph_0) = 2^{\aleph_0}$.

Back to T:

$$2^{\aleph_0} = I(Th(\omega + \omega^*, <), \aleph_0) = I((T_{\bar{c}})_\phi, \aleph_0) = I(T_{\bar{c}}, \aleph_0) = I(T, \aleph_0)$$

Recall: $Th(\omega + \omega^*, <)$ is Borel complete.

Question

If T has a definable (or interpretable) infinite discrete linear order, must T be Borel complete?

Possible proof scheme for T: Prove that "$= 2^{\aleph_0}$" can be replaced by "Borel complete" in each of the following:

1. If T is not small, then $I(T, \aleph_0) = 2^{\aleph_0}$;
2. (T small) $I(T_{\bar{c}}, \aleph_0) = 2^{\aleph_0} \Rightarrow I(T, \aleph_0) = 2^{\aleph_0}$;
3. (T small) $I(T_{\phi}, \aleph_0) = 2^{\aleph_0} \Rightarrow I(T, \aleph_0) = 2^{\aleph_0}$.

All open;

For general T, it is known that in (1) we cannot make it: there are non-small theories that are not Borel complete, but

The space of L-structures

Let L be a countable language.

$$\mathbb{X}_L \text{ -- the space of all } L\text{-structures with domain } \omega$$

$V_{\phi(\bar{n})} = \{M \in \mathbb{X}_L \mid M \models \phi(\bar{n})\}$ the basic clopen sets.

1. $\mathbb{X}_L$ is a standard Borel space; usually, $\mathbb{X}_L = 2^{\prod_{R \in L} \omega^{ar(R)}}$ (such that a graph (ω, R) is identified with $R \in 2^{\omega \times \omega}$);
2. $Mod(\varphi) = \{M \in \mathbb{X}_L \mid M \models \varphi\}$ is a invariant Borel subset of $\mathbb{X}_L$ for all $\varphi \in L_{\omega_1,\omega}$;
3. $\cong_\varphi$ is a invariant Σ^1_1-subset of $Mod(\varphi) \times Mod(\varphi)$;
4. Every $\cong_\varphi$-class is Borel.

Borel reducibility

1. Let E, F be equivalence relations on standard Borel spaces X, Y; E is Borel reducible to F, denoted by $E \leqslant_B F$, if there is a Borel map $f : X \to Y$ such that $xEy \Leftrightarrow f(x)Ff(y)$.
2. For $\psi \in L_{\omega_1,\omega}$ and $\phi \in L'_{\omega_1,\omega}$ define $\phi \leqslant_B \psi$ if and only if $(Mod(\phi), \cong_\phi) \leqslant_B (Mod(\psi), \cong_\psi)$.
3. $\phi \in L_{\omega_1,\omega}$ is Borel-complete if $\psi \leqslant_B \phi$ holds for all $\psi \in L'_{\omega_1,\omega}$ (in any countable L').
4. A class of structures is Borel complete if it is axiomatized by a Borel complete sentence.

Theorem (Friedman, Stanley (1989))

The following classes are Borel complete: graphs, groups, linear orders, trees.

Question 2 is a part of:

Question (Laskowski)

Can Borel completeness be gained or lost by naming a constant?

Theorem (Rast)

$T_{\bar{c}}$ is Borel iff T is Borel.