Ideal topological spaces
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We can say that ideals are folklore in mathematics. If \( X \) is a nonempty set, a family \( \mathcal{I} \subset P(X) \) satisfying

(I0) \( \emptyset \in \mathcal{I} \),
(I1) If \( A \in \mathcal{I} \) and \( B \subseteq A \), then \( B \in \mathcal{I} \),
(I2) If \( A, B \in \mathcal{I} \), then \( A \cup B \in \mathcal{I} \),

is called an **ideal** on \( X \).

\{\emptyset\}, \( P(X) \),
principal ideal - \( A \downarrow \),
finite sets - \( \text{Fin} \),
countable sets - \( \mathcal{I}_{\text{count}} \),
cardinality less than \( \kappa \) - \( \mathcal{I}_{<\kappa} \),
closed and discrete sets - \( \mathcal{I}_{\text{cd}} \),
scattered sets (with \( T_1 \)) - \( \mathcal{I}_{\text{sc}} \),
relatively compact sets - \( \mathcal{I}_{K} \),
nowhere dense sets - \( \mathcal{I}_{\text{nwd}} \),
meager sets - \( \mathcal{I}_{\text{mgr}} \),
sets of measure zero - \( \mathcal{I}_{m0} \).

No need to define topological space \( \langle X, \tau \rangle \).
Triplet \( \langle X, \tau, \mathcal{I} \rangle \) is called **ideal topological space**.
Short history

The first steps in introducing topological spaces enhanced by an ideal is due to Kuratowski \cite{4, 5} in 1933, who introduced local function as a generalization to closure.

A little bit later ideals in topological spaces were studied by Vaidyanathaswamy \cite{10} (1944).

1958. Freud \cite{2} generalized Cantor-Bendixson theorem using ideal topological space.

1971. Scheinberg \cite{9} applied ideals in the measure theory.

In 1990 Janković and Hamlett \cite{3} wrote a survey paper on the topic of ideal topological spaces.

Today this paper is a starting point, and a pattern for introducing many variations and generalizations of open sets defined by ideals.
Local function

Definition 1 (Kuratowski 1933) [4] Let \( (X, \tau, \mathcal{I}) \) be an ideal topological space. Then

\[
A^*(\mathcal{I}, \tau) = \{ x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x) \}
\]

is called the local function of \( A \) with respect to \( \mathcal{I} \) and \( \tau \).

For \( \mathcal{I} = \{ \emptyset \} \) we have that \( A^*(\mathcal{I}, \tau) = \text{Cl}(A) \).

For \( \mathcal{I} = P(X) \) we have that \( A^*(\mathcal{I}, \tau) = \emptyset \).

For \( \mathcal{I} = \text{Fin} \) we have that \( A^*(\mathcal{I}, \tau) \) is the set of \( \omega \)-accumulation points of \( A \).

For \( \mathcal{I} = \mathcal{I}_{\text{count}} \) we have that \( A^*(\mathcal{I}, \tau) \) is the set of condensation points of \( A \).

The local function has the following properties (see [3]):

1. \( A \subseteq B \Rightarrow A^* \subseteq B^* \);
2. \( A^* = \text{Cl}(A^*) \subseteq \text{Cl}(A) \);
3. \( (A^*)^* \subseteq A^* \);
4. \( (A \cup B)^* = A^* \cup B^* \);
5. If \( I \in \mathcal{I} \), then \( (A \cup I)^* = A^* = (A \setminus I)^* \).
"Idealized" topology

**Definition 2** \( \text{Cl}^*(A) = A \cup A^* \) is a Kuratowski closure operator, i.e.

1. \( \text{Cl}^*(\emptyset) = \emptyset \),
2. \( A \subseteq \text{Cl}^*(A) \),
3. \( \text{Cl}^*(A \cup B) = \text{Cl}^*(A) \cup \text{Cl}^*(B) \), and
4. \( \text{Cl}^*(\text{Cl}^*(A)) = \text{Cl}^*(A) \).

And therefore it generates a topology on \( X \)

\[ \tau^*(\mathcal{I}) = \{A : \text{Cl}^*(X \setminus A) = X \setminus A\} \]

\[ \tau \subseteq \tau^* \subseteq P(X) \]

Set \( A \) is closed in \( \tau^* \) iff \( A^* \subseteq A \).

If \( \Psi(A) = X \setminus (X \setminus A)^* \), then set \( O \in \tau^* \) iff \( O \subseteq \Psi(O) \).

\[ \beta(\mathcal{I},\tau) = \{V \setminus I : V \in \tau, I \in \mathcal{I}\} \] is a basis for \( \tau^* \)

\[ \tau^* = \tau^{**} \]
For $\mathcal{I} = \{\emptyset\}$ we have that $\tau^*(\mathcal{I}) = \tau$.

For $\mathcal{I} = P(X)$ we have that $\tau^*(\mathcal{I}) = P(X)$.

If $\mathcal{I} \subseteq \mathcal{J}$ then $\tau^*(\mathcal{I}) \subseteq \tau^*(\mathcal{J})$.

If $Fin \subseteq \mathcal{I}$ then $\langle X, \tau^* \rangle$ is $T_1$ space.

If $\mathcal{I} = Fin$, then $\tau_{ad}(\mathcal{I})$ is the cofinite topology on $X$.

If $\mathcal{I} = \mathcal{I}_{m0}$ - ideal of the sets of measure zero, then $\tau^*$-Borel sets are precisely the Lebesgue measurable sets. (Scheinberg 1971)[9]

For $\mathcal{I} = \mathcal{I}_{nwd}$ then $A^* = \text{Cl}(\text{Int}(\text{Cl}(A)))$ and $\tau^*(\mathcal{I}_{nwd}) = \tau^\alpha$. ($\alpha$-open sets, $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ - (Njástad 1965)[6])
Let \( \langle X, \tau, I \rangle \) be an ideal topological space. We say \( \tau \) is compatible with the ideal \( I \), denoted \( \tau \sim I \) if the following holds for every \( A \subseteq X \): if for every \( x \in A \) there exists a \( U \in \tau(x) \) such that \( U \cap A \in I \), then \( A \in I \).

\[
\tau \sim I \text{ implies } \beta = \tau^*
\]

\[
\tau \sim I \text{ iff } A \setminus A^* \in I, \text{ for each } A.
\]

**Theorem 1** \( \langle X, \tau \rangle \) is hereditarily Lindelöf iff \( \tau \sim I_{\text{count}} \)

**Theorem 2** \( \tau \sim I_{\text{nwd}} \)

**Theorem 3** \( \tau \sim I_{\text{mgr}} \)

**Theorem 4** Let \( \langle X, \tau, I \rangle \) be an ideal topological space. The following are equivalent.

(a) \( I \sim \tau \) and \( \text{Fin} \subseteq I \).

(b) Scattered sets in \( \langle X, \tau^* \rangle \) are in \( I \).

(c) Discrete sets in \( \langle X, \tau^* \rangle \) are in \( I \).
Theorem 5 (Cantor-Bendixson). A second countable (moreover, hereditarily Lindelof) space can be represented as the union of two sets, one of which is perfect (closed without isolated points) and the other countable.

Theorem 6 (Freud 1958)[2] Let \( \langle X, \tau, \mathcal{I} \rangle \) be an ideal topological space such that \( \mathcal{I} \sim \tau \) and \( Fin \subseteq \mathcal{I} \). If a set \( A \) is closed with respect to \( * \), then \( A \) is the union of a set which is perfect with respect to \( \tau \) and a set in \( \mathcal{I} \).
\( X = X^* \)

**Theorem 7** (Samules 1975)\[8\] Let \( \langle X, \tau, \mathcal{I} \rangle \) be an ideal topological space. Then \( X = X^* \) iff \( \tau \cap \mathcal{I} = \{\emptyset\} \).

\( \tau_s \) is the family of regular open sets (\( U = \text{Int}(\text{Cl}(U)) \)) in \( \tau \)

**Theorem 8** (Janković Hamlett 1990)\[3\] Let \( \langle X, \tau \rangle \) be a space with an ideal \( \mathcal{I} \) on \( X \). If \( X = X^* \) then \( \tau_s = \tau_s^* \).

It was observed in (Bourbaki 1966)\[1\] that some important topological properties are shared by \( \langle X, \tau \rangle \) and \( \langle X, \tau_s \rangle \). Some of these properties, so called semiregular properties, are: Hausdorffness, property of a space being Urysohn \( (T_{2\frac{1}{2}}) \), connectedness, extremal disconnectedness, H-closedness, light compactness (every locally finite collection of open subsets is finite), pseudocompactness, . . .

**Theorem 9** Semiregular properties are shared by \( \langle X, \tau \rangle \) and \( \langle X, \tau^* \rangle \) if \( X = X^* \)
A space \( \langle X, \tau \rangle \) is said to be a **Baire space** if the intersection of every countable family of open dense sets in \( \langle X, \tau \rangle \) is dense.

\[
\langle X, \tau \rangle \text{ is a Baire space iff } X = X^*(I_{mgr}).
\]

Space is anticomponent iff the only compact sets are finite.

If \( \langle X, \tau \rangle \) is Hausdorff, then \( \tau^*(I_{\text{count}}), \tau^*(I_{\text{sc}}) \) and \( \tau^*(I_K) \) are anticomponent.

If \( \text{Fin} \subseteq I \) and \( \tau \sim I \) then also \( \tau^*(I) \) is anticomponent (Janković Hamlett 1990)[3].

**Theorem 10 (Samuels 1971)[3]** If \( X = X^* \) and \( Y \) is regular, then \( f : \langle X, \tau \rangle \rightarrow Y \) is continuous iff \( f : \langle X, \tau^* \rangle \rightarrow Y \) is continuous.
References


