Preserving continuity in Ideal topological spaces

Anika Njamcul and Aleksandar Pavlović

Fast Introduction

Definition 1. (Kuratowski 1933)[3] Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then

$$A^*(\mathcal{I}, \tau) = \{x \in X : A \cap U \notin \mathcal{I} \text{ for every } U \in \tau(x)\}$$

is called the **local function** of A with respect to $\mathcal{I}$ and τ.

$\mathrm{Cl}^*(A) = A \cup A^*$

$\tau^*(\mathcal{I}) = \{A : \mathrm{Cl}^*(X \setminus A) = X \setminus A\}.$

F is closed iff $F^* \subseteq F$

$\psi(A) = X \setminus (X \setminus A)^*$

U is open iff $U \subseteq \psi(U)$

$\psi(\tau) = \{\psi(U) : U \in \tau\}.$

$\psi(\tau) \subseteq \tau \subseteq \tau^*$

Previous results

Theorem 1. *([7] Samuels 1971.) If* $X = X^*$ $(\mathcal{I} \cap \tau = \{\emptyset\})$ *and* Y *is regular then* $f : \langle X, \tau \rangle \to Y$ *is continuous iff* $f : \langle X, \tau^* \rangle \to Y$ *is continuous*

Theorem 2. *([4] Natkaniec 1986.) Let* $f : X \to \mathbb{R}$*, where* X *is a Polish space with topology* τ*, and* $\mathcal{I}$ *a* σ*-complete ideal on* X *such that* $Fin \subset \mathcal{I}$ *and* $\mathcal{I} \cap \tau = \{\emptyset\}$*.*

If $f : \langle X, \tau^* \rangle \to \langle R, \mathcal{O}_{nat} \rangle$ *is a continuous function, then* $f : \langle X, \tau \rangle \to \langle R, \mathcal{O}_{nat} \rangle$ *is also continuous.*

$\langle X, \tau, \mathcal{I} \rangle$ is $\mathcal{I}$-compact ([5, 6] Newcomb 1968., Rančin 1972.) iff for each open cover $\{U_\lambda : \lambda \in \Lambda\}$ exists finite subcollection $\{U_{\lambda_k} : k \leq n\}$ such that $X \setminus \bigcup \{U_{\lambda_k} : k \leq n\} \in \mathcal{I}$.

Theorem 3. *([1] Hamlett, Janković 1990.) Let* $f : \langle X, \tau, \mathcal{I} \rangle \to \langle Y, \sigma, f[\mathcal{I}] \rangle$ *be a bijection such that* $\langle X, \tau \rangle$ *is* $\mathcal{I}$*-compact and* $\langle Y, \sigma \rangle$ *is Hausdorff. If* $f : \langle X, \tau^* \rangle \to \langle Y, \sigma \rangle$ *is continuous, then* $f : \langle X, \tau^* \rangle \to \langle Y, \sigma^* \rangle$ *is a homeomorphism.*

Theorem 4. *([2] Hamlett, Rose 1990.) Let* $\langle X, \tau, \mathcal{I}\rangle, \langle Y, \sigma, \mathcal{J}\rangle$ *be ideal topological spaces. Let* $f : \langle X, \tau\rangle, \langle Y, \langle\psi(\sigma)\rangle\rangle$ *be a continuous injection,* $\mathcal{J} \sim \sigma$ *and* $f^{-1}[\mathcal{J}] \subset \mathcal{I}$*. Then* $\psi(f[A]) \subseteq f[\psi(A)]$*, for each* $A \subseteq X$*.*

Theorem 5. *([2] Hamlett, Rose 1990.) Let* $\langle X, \tau, \mathcal{I}\rangle, \langle Y, \sigma, \mathcal{J}\rangle$ *be ideal topological spaces. Let* $f : \langle X, \langle\psi(\tau)\rangle\rangle \to \langle Y, \sigma\rangle$ *be an open bijection,* $\mathcal{I} \sim \tau$ *and* $f[\mathcal{I}] \subset \mathcal{J}$*. Then* $f[\psi(A)] \subseteq \psi(f[A])$*, for each* $A \subseteq X$*.*

Theorem 6. *([2] Hamlett, Rose 1990.) Let* $\langle X, \tau, \mathcal{I}\rangle \to \langle Y, \sigma, \mathcal{J}\rangle$ *be ideal topological spaces. Let* $f : X \to Y$ *be a bijection and* $f[\mathcal{I}] = \mathcal{J}$*. Then the following conditions are equivalent*

a) $f : \langle X, \tau^*\rangle \to \langle Y, \sigma^*\rangle$ *is a homeomorphism;*

b) $f[A^*] = (f[A])^*$*, for each* $A \subseteq X$*;*

c) $f[\psi(A)] = \psi(f[A])$*, for each* $A \subseteq X$*.*

Results

Theorem 7. *Let* $\langle X, \tau_X, \mathcal{I}_X \rangle$ *and* $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ *be ideal topological spaces. If* $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ *is a continuous function and for all* $I \in \mathcal{I}_Y$ *we have* $f^{-1}[I] \in \mathcal{I}_X$*. Then there hold the following equivalent conditions:*

a) $\forall A \subseteq X \ f[A^*] \subseteq (f[A])^*$;

b) $\forall B \subseteq Y \ (f^{-1}[B])^* \subseteq f^{-1}[B^*]$.

Proof. Let us prove a).

Suppose $\exists A \subseteq X$ and $y \in f[A^*] \setminus (f[A])^*$.

Let $x \in A^*$ such that $f(x) = y$.

$$\forall U \in \tau_X(x) \ U \cap A \notin \mathcal{I}_X. \tag{1}$$

$y \notin (f[A])^*$

$\exists V \in \tau_Y(y)$ such that $V \cap f[A] \in \mathcal{I}_Y$.

$f^{-1}[V \cap f[A]] \in \mathcal{I}_X$.

$f^{-1}[V] \cap f^{-1}[f[A]] \in \mathcal{I}_X$,

From $A \subseteq f^{-1}[f[A]]$, we have

$$f^{-1}[V] \cap A \in \mathcal{I}_X. \tag{2}$$

Due the continuity of f, $f^{-1}[V] \in \tau_X(x)$

(2) contradicts (1), proving a).

a) $\forall A \subseteq X\ f[A^*] \subseteq (f[A])^*$;
b) $\forall B \subseteq Y\ (f^{-1}[B])^* \subseteq f^{-1}[B^*]$.

(a) is equivalent to b)).
Suppose a) holds
Let $B \subseteq Y$. Then $f[(f^{-1}[B])^*] \subseteq (f[f^{-1}[B]])^*$
$f[(f^{-1}[B])^*] \subseteq B^*$.
$f^{-1}[f[(f^{-1}[B])^*]] \subseteq f^{-1}[B^*]$
$(f^{-1}[B])^* \subseteq f^{-1}[B^*]$.

Suppose b) holds
Let $A \subseteq X$.
$f^{-1}[(f[A])^*] \supseteq (f^{-1}[f[A]])^* \supseteq A^*$.
$f[f^{-1}[(f[A])^*]] \supseteq f[A^*]$.
$(f[A])^* \supseteq f[A^*]$. $\square$

Example 1. The opposite does not hold even
if for each $I \in \mathcal{I}_Y$ holds $f^{-1}[I] \in \mathcal{I}_X$.
$\mathcal{I}_X = P(X)$, $A^* = \emptyset$, $f^{-1}[I] \in \mathcal{I}_X = P(X)$.
But f does not have to be continuous, in general.

Theorem 8. *Let* $\langle X, \tau_X, \mathcal{I}_X\rangle$ *and* $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ *be ideal topological spaces. If* $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ *is a continuous function and for all* $I \in \mathcal{I}_Y$ *we have* $f^{-1}[I] \in \mathcal{I}_X$. *Then there hold the following three equivalent conditions:*

a) $\forall A \subseteq X \ f[\overline{A}^{\tau_X^*}] \subseteq \overline{f[A]}^{\tau_Y^*}$;

b) $\forall B \subseteq Y \ \overline{(f^{-1}[B])}^{\tau_X^*} \subseteq f^{-1}[\overline{B}^{\tau_Y^*}]$;

c) $f : \langle X, \tau_X^*\rangle \to \langle Y, \tau_Y^*\rangle$ *is a continuous function;*

Proof. Proving a): $f[\overline{A}^{\tau_X^*}] = f[A \cup A^*] = f[A] \cup f[A^*] \subseteq f[A] \cup (f[A])^* = \overline{f[A]}^{\tau_Y^*}$. □

Example 2. Condition a) in Theorem 7 is not equivalent to the continuity of $f : \langle X, \tau_X^*\rangle \to \langle Y, \tau_Y^*\rangle$.

If $\tau_X = P(X)$, then each mapping is continuous. Let $x \in X$.
Let $\{x\} \notin \mathcal{I}_X$ and, for $y = f(x)$,$\{y\} \in \mathcal{I}_Y$
Then $x \in \{x\}^* = \{x\}$, so $y \in f[\{x\}^*] = \{y\}$.
But $U \cap f[\{x\}] = \{y\} \in \mathcal{I}_Y$, for each $U \in \tau_Y(y)$,
so $y \notin (f[\{x\}])^*$,
so condition a) does not hold.

If we add that f is a bijection, we obtain the following result.

Theorem 9. *Let $\langle X, \tau_X, \mathcal{I}_X\rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y\rangle$ be ideal topological spaces. If $f : \langle X, \tau_X\rangle \to \langle Y, \tau_Y\rangle$ is a continuous bijection and for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$. Then there hold the following equivalent conditions:*
a) $\forall A \subseteq X\ \Psi(f[A]) \subseteq f[\Psi(A)]$; *b)* $\forall B \subseteq Y\ f^{-1}[\Psi(B)] \subseteq \Psi(f^{-1}[B])$.

Proof. (a) $\Rightarrow$ b)) $B \subseteq Y$. $f^{-1}[\Psi(B)] = f^{-1}[\Psi(f[f^{-1}[B]])] \subseteq f^{-1}[f[\Psi(f^{-1}[B])]] = \Psi(f^{-1}[B])$.
(b) $\Rightarrow$ a)) $A \subseteq X$. $\Psi(f[A]) = f[f^{-1}[\Psi(f[A])]] \subseteq f[\Psi(f^{-1}[f[A]])] = f[\Psi(A)]$.
(a)) Suppose that there exists $A \subseteq X$ such that
$\Psi(f[A]) \setminus f[\Psi(A)] \neq \emptyset$.

$$
\begin{aligned}
(3) \qquad & \Psi(f[A]) \setminus f[\Psi(A)] \\
&= (Y \setminus (Y \setminus f[A])^*) \setminus f[X \setminus (X \setminus A)^*] \\
&= (Y \setminus (Y \setminus f[A])^*) \setminus (f[X] \setminus f[(X \setminus A)^*]) \\
&= (Y \setminus (Y \setminus f[A])^*) \setminus (Y \setminus f[(X \setminus A)^*]) \\
&= f[(X \setminus A)^*] \setminus (Y \setminus f[A])^* \\
\text{(by surjection)} &= f[(X \setminus A)^*] \setminus (f[X] \setminus f[A])^* \\
\text{(by injection)} &= f[(X \setminus A)^*] \setminus (f[X \setminus A])^* \neq \emptyset,
\end{aligned}
$$

but this contradicts condition a) from Theorem 7. □

Remark 1. This is also a proof that, if f is bijection, conditions a) and b) from the previous theorem and from Theorem 7 are equivalent. Just the set $B \subset X$ which possibly violates condition a) from Theorem 7 write in form of $X \setminus A$ and apply (3).

Conjecture (20.3.2021.): If for all $I \in \mathcal{I}_Y$ we have $f^{-1}[I] \in \mathcal{I}_X$ then we also have equivalence.

Now, we will consider open mappings.

Theorem 10. *Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is an open function and for all $I \in \mathcal{I}_X$ we have $f[I] \in \mathcal{I}_Y$. Then there hold the following equivalent conditions:*

a) $\forall A \subseteq X \ \ f[\Psi(A)] \subseteq \Psi(f[A])$;
b) $\forall B \subseteq Y \ \ \Psi(f^{-1}[B]) \subseteq f^{-1}[\Psi(B)]$;

Theorem 11. *Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is an open function and for all $I \in \mathcal{I}_X$ we have $f[I] \in \mathcal{I}_Y$. Then $f : \langle X, \tau_X^* \rangle \to \langle Y, \tau_Y^* \rangle$ is an open function.*

Theorem 12. *Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is an open bijection and for all $I \in \mathcal{I}_X$ we have $f[I] \in \mathcal{I}_Y$. Then there hold the following equivalent conditions:*

a) $\forall A \subseteq X \ \ (f[A])^* \subseteq f[A^*]$;
b) $\forall B \subseteq Y \ \ f^{-1}[B^*] \subseteq (f^{-1}[B])^*$.

Corollary 1. ([2] Hamlett, Rose 1990.) Let $\langle X, \tau_X, \mathcal{I}_X \rangle$ and $\langle Y, \tau_Y, \mathcal{I}_Y \rangle$ be ideal topological spaces. If $f : \langle X, \tau_X \rangle \to \langle Y, \tau_Y \rangle$ is homeomorphism and for each $I \subset X$ there holds $I \in \mathcal{I}_X$ iff $f[I] \in \mathcal{I}_Y$. Then the following equivalent conditions hold:

a) $f : \langle X, \tau_X^* \rangle \to \langle Y, \tau_Y^* \rangle$ is a homeomorphism;
b) $\forall A \subseteq X \ \ (f[A])^* = f[A^*]$;
c) $\forall B \subseteq Y \ \ f^{-1}[B^*] = (f^{-1}[B])^*$.

References

[1] Hamlett, T. R., and Janković, D. Compactness with respect to an ideal. *Boll. Un. Mat. Ital. B (7) 4*, 4 (1990), 849–861.

[2] Hamlett, T. R., and Rose, D. *-topological properties. *Internat. J. Math. Math. Sci. 13*, 3 (1990), 507–512.

[3] Kuratowski, K. *Topologie I.* Warszawa, 1933.

[4] Natkaniec, T. On I-continuity and I-semicontinuity points. *Mathematica Slovaca 36*, 3 (1986), 297–312.

[5] Newcomb, Jr, R. L. *Topologies which are compact modulo an ideal.* ProQuest LLC, Ann Arbor, MI, 1968. Thesis (Ph.D.)–University of California, Santa Barbara.

[6] Rančin, D. V. Compactness modulo an ideal. *Dokl. Akad. Nauk SSSR 202* (1972), 761–764.

[7] Samuels, P. A topology formed from a given topology and ideal. *J. London Math. Soc. (2) 10*, 4 (1975), 409–416.