Representing inverse semigroups in complete inverse algebras

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Figure: Rocky Cape

Figure: Zabranjeno plivanje!

Figure: After the flood

OUTLINE

Inverse semigroups

Representations of inverse semigroups

Inverse Algebras

Boolean inverse algebras/semigroups

Studying reps, using atoms

Generic examples of inverse semigroups I

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- there is a pseudoproduct $(D, f, R) \otimes (D', g, R')\colon = (\ \cdot\ , f|_{R \cap D'} \circ {}_{R \cap D'}|g,\ \cdot\ )$ which is total (defined for all pairs)

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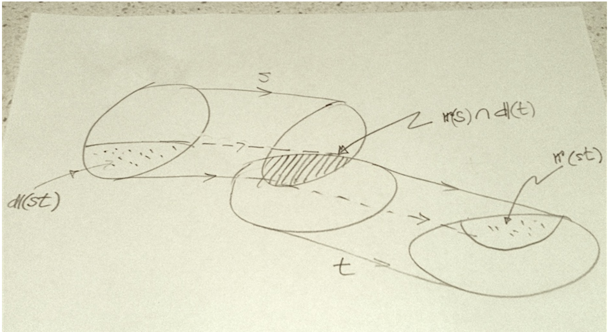

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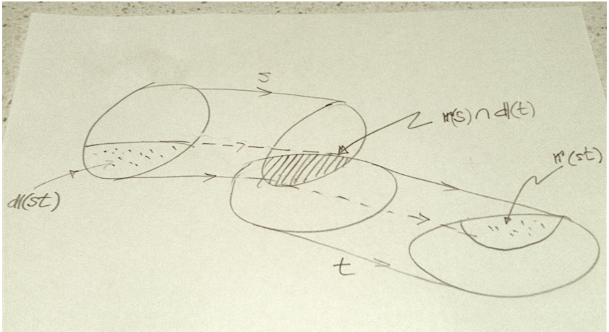

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- Extend to partial automorphisms of algebras, spaces, etc.

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- This is the *dual* symmetric inverse monoid

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- The respective semilattices-of-idempotents have very special structures—they are the power set 2^X and the (set-) partition lattice $\mathscr{P}(X)$.

Other inverse semigroups

Obviously this also works for a wide class of objects (anything with a notion of subobject or quotient object), giving inverse semigroups of partial isomorphisms or of bicongruences of:

- vector spaces
- topological spaces
- graphs
- groups

which in some special cases determine the object

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- Books of MV Lawson, M Petrich

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- Pultr & Trnkova book; algebraic universality property

Transformation

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Figure: Domain: Cumquat bush

Transformation

▶ 

Figure: Range: Marmalade

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- —use trick, and note action fixes $\emptyset, X$

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- . . . and these are best possible.

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- –especially for a *wide* S with relatively many idempotent atoms compared to its height

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- every *transitive* one has an 'internal' description in terms of appropriately defined *cosets* of closed inverse subsemigroups
- But what about reps in $\mathscr{I}_X^*$?

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- S is an *inverse* $\vee$*-semigroup* if any compatible set has a join

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- $\bigvee X = \left(\bigvee_{x \in X} xx^{-1}\right) u = u \left(\bigvee_{x \in X} x^{-1}x\right)$

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- (*Ehresmann's lemma*)

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- (Note, the calculations are in A, not necessarily in X. And *bounded above in A* may be replaced by *compatible* for the pair or subset.)

Distributive and Boolean inverse algebras

- A subset X of A is *distributive* if $x(y \vee z) = xy \vee xz$ for all $x, y, z \in X$ with y, z bounded above in A, and
- *completely distributive* if $x(\bigvee_{y \in Y} y) = \bigvee_{y \in Y} xy$ for all $x \in X$ and all $Y \subseteq X$ such that Y has an upper bound in A.

- A is *Boolean* if $E(A)$ is boolean.

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- is not Boolean but I think it is still special !

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- For a Boolean A, being atomistic is equivalent to being *atomic*, that is, each element is above an atom.

More on atoms

- Let A be a complete atomistic inverse algebra, with its set of primitive idempotents (atoms of $E(A)$) denoted by $P = P(A)$. Write $P^0 = P \cup \{0\}$.

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Studying representations

A simplification: To avoid writing $\phi\colon S' \to A$ we consider how $S'\phi = S$ sits in A. (The congruences on S are well-described.)

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- $\mathscr{T} = \mathscr{T}_S$ is an equivalence on its domain $\subset P$

More on atoms

A side-trip, useful technically: The Following Are Equivalent:

- $q = s^{-1}ps$;

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- Any *effective* S in $\mathscr{I}_X$
- decomposes to a 'sum' of *transitive* ones
- each of which uses one orbit

Classifying representations / subsemigroups

Recall the 'classical' case:

- every transitive one has an 'internal' description in S

Effective; transitive

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- So say that $S \leq A$ is *weakly effective* if the only local algebra containing S is A itself: $S \leq eAe$ implies $e = 1$. ($s = se = es$ for all $s \in S \Rightarrow e = 1_A$.)

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Transitivity

- Classically: $S \leq \mathscr{I}_X$ is transitive if, given any $x, y \in X$, there is $s \in S$ with $(x, y) \in s$. $((x, x) \mapsto (y, y))$

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- abstract version: S is *strongly transitive* in A if there is only one orbit of the action, i.e., each atom of A is underneath some element of S
- implications for the structure of A:
  ... all atoms of A form one $\mathscr{D}$-class. Too strong?

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- S is *weakly transitive* if $\mathscr{T}_S$ has just one class (AND not necessarily all of P). That is, for each pair $p, q \in P$ *such that* $pS \neq \{0\}$ *and* $qS \neq \{0\}$, $p = s^{-1}qs$ for some $s \in S$.

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- So 'weakly effective and transitive' means both are weak-sense
- We also have to give something away in the component maps: say that ϕ is a *lax* homomorphism if $(st)\phi \leq (s\phi)(t\phi)$

Theorems

- Any (effective) representation of an inverse semigroup S in a complete atomistic inverse algebra A is equivalent to a product of weakly transitive effective lax representations of S.

Theorems

- Any effective representation of an inverse semigroup S in a complete atomistic distributive inverse algebra A is equivalent to a sum of transitive effective representations of S.

Theorems

- Any effective representation of an inverse semigroup S in a complete atomic Boolean inverse algebra A is equivalent to an orthogonal sum of transitive effective representations of S.

Theorems

- Any effective representation of an inverse semigroup S in a matroid inverse algebra A is equivalent to a product of transitive and effective representations of S.

Theorems need definitions!

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- If L is a matroid lattice, then it is meet-continuous.

Key methods

- Let the blocks of $\mathscr{T}_S$ be $\{P_i \colon i \in I\}$ for some index set I.

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