Topological finiteness properties of monoids

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The word problem for monoids and groups

Definition

A monoid M with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^*$ decides whether or not they represent the same element of M.

Example. $M \cong \langle a, b \mid ab = ba \rangle$ has decidable word problem.

Some history

- Markov (1947) and Post (1947): first examples of finitely presented monoids with undecidable word problem;
- Turing (1950): finitely presented cancellative semigroup with undecidable word problem;
- Novikov (1955) and Boone (1958): finitely presented group with undecidable word problem.

Complete rewriting systems

A - alphabet, $R \subseteq A^* \times A^*$ - rewrite rules, $\langle A \mid R \rangle$ - rewriting system
Write $r = (r_{+1}, r_{-1}) \in R$ as $r_{+1} \to r_{-1}$.

Define a binary relation $\to_R$ on A^* by

$$u \to_R v \iff u \equiv w_1 r_{+1} w_2 \text{ and } v \equiv w_1 r_{-1} w_2$$

for some $(r_{+1}, r_{-1}) \in R$ and $w_1, w_2 \in A^*$.

$\xrightarrow{*}_R$ is the transitive and reflexive closure of $\to_R$

Noetherian: No infinite descending chain

$$w_1 \to_R w_2 \to_R \cdots \to_R w_n \to_R \cdots$$

Confluent: Whenever

$$u \xrightarrow{*}_R v \text{ and } u \xrightarrow{*}_R v'$$

there is a word $w \in A^*$:

$$v \xrightarrow{*}_R w \text{ and } v' \xrightarrow{*}_R w$$

Definition: $\langle A \mid R \rangle$ is a finite complete rewriting system if it is complete (noetherian and confluent) and $|A| < \infty$ and $|R| < \infty$.

Complete rewriting systems

Example (Free commutative monoid)

$\langle a, b \mid ba \to ab \rangle$

Normal forms (irreducibles) = $\{a^i b^j : i, j \geq 0\}$

Example (Free group)

$\langle a, a^{-1}, b, b^{-1} \mid aa^{-1} \to 1,\ a^{-1}a \to 1,\ bb^{-1} \to 1,\ b^{-1}b \to 1 \rangle.$

Normal forms (irreducibles) = { freely reduced words }.

Important basic fact

If a monoid M admits a presentation by a finite complete rewriting system, then M has decidable word problem.

The homological finiteness property FP_n

$\mathbb{Z}M$ - integral monoid ring, e.g. $4m_1 - 2m_2 + 3m_3 \in \mathbb{Z}M$

Definition

A monoid is of type left-FP_n if $\mathbb{Z}$ has a free resolution as a trivial left $\mathbb{Z}M$-module that is finite through dimension n. i.e. there is a sequence:

$$F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

such that for all i we have:

- F_i is a finitely generated free left $\mathbb{Z}M$-module i.e.

$$F_i \cong \mathbb{Z}M \oplus \mathbb{Z}M \oplus \cdots \oplus \mathbb{Z}M$$

- ∂_i is a homomorphism, and the sequence is exact, i.e.
  - $\mathrm{im}(\partial_i) = \ker(\partial_{i-1})$ and $\mathrm{im}(\partial_0) = \mathbb{Z}$.

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For any monoid:

- finitely generated $\Rightarrow$ left-FP_1, finitely presented $\Rightarrow$ left-FP_2
- **Anick (1986):** If a monoid M is presented by a finite complete rewriting system then M is of type left-FP_∞.

One-relation monoids

Longstanding open problem

Is the word problem decidable for one-relation monoids $\langle A \mid u = v \rangle$?

Related open problem

Does every one-relation monoid $\langle A \mid u = v \rangle$ admit a presentation by a finite complete rewriting system?

If yes then every one-relation monoid would be of type left-FP_∞ so we ask:

Question: Is every one-relator monoid $\langle A \mid u = v \rangle$ of type left-FP_∞?

Magnus (1932): Proved that one-relator groups have decidable word problem.

Cohen–Lyndon (1963): Shows that every one-relator group is of type FP_∞.

The topological finiteness property F_n

Definition (C. T. C. Wall (1965))

A $K(G,1)$-complex is a CW complex with fundamental group G and all other homotopy groups trivial (i.e. the space is aspherical). A group G is of type F_n $(0 \leq n < \infty)$ if there is a $K(G,1)$-complex with finite n-skeleton

For any group:

(i) $\mathrm{F}_1 \equiv$ finitely generated, $\mathrm{F}_2 \equiv$ finite presented.

(ii) $\mathrm{F}_n \Rightarrow \mathrm{FP}_n$

(iii) For finitely presented groups $\mathrm{F}_n \equiv \mathrm{FP}_n$.

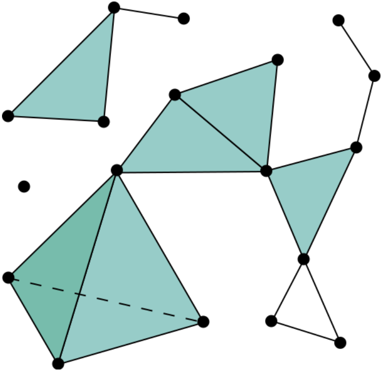
Aim

Develop a theory of topological finiteness properties of monoids. A good definition of F_n for monoids should satisfy (ii), so that it can be used to study FP_n.

Cell complexes

...spaces that can be decomposed nicely into a disjoint union of cells

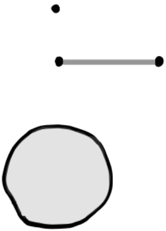
- $I = [0,1] \subseteq \mathbb{R}$ - unit interval
- S^n - unit sphere in $\mathbb{R}^{n+1}$
  = all points at distance 1 from the origin.
- B^n - closed unit ball in $\mathbb{R}^n$ = all points of distance ≤ 1 from the origin.
- $\partial B^n = S^{n-1}$ = the boundary of the n ball.
- e^n - an n-cell, homeomorphic to the open n ball $B^n - \partial B^n$.



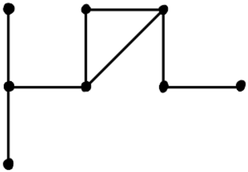
0 - cell

1 - cell

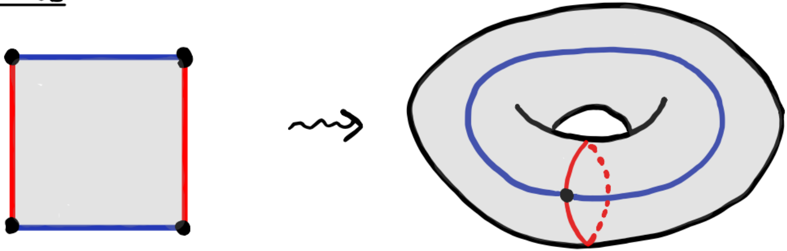
2 - cell



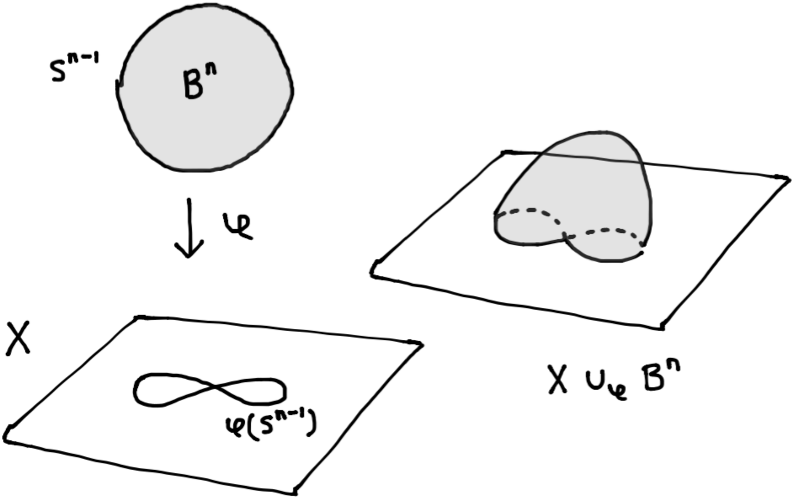
Graphs



Torus



Attaching an n-cell



CW complex definition

Definition

A CW decomposition of a topological space X is a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

such that

- X_0 is discrete set, whose points are regarded as 0 cells
- The n-skeleton X_n is obtained from X_{n-1} by attaching a (possibly) infinite number of n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X_{n-1}$.
- We have $X = \cup X_n$ with the *weak* topology (this means that a set $U \subseteq X$ is open if and only if $U \cap X_n$ is open in X_n for each n).

A CW complex[2] is a space X equipped with a CW decomposition.

[2] C stands for 'closure-finite', and the W for 'weak topology'.

$K(G,1)$ of a group and property F_n

Definition

A $K(G,1)$-complex is a CW complex with fundamental group G and all other homotopy groups trivial (i.e. the space is aspherical).

Existence: Every group G admits a $K(G,1)$-complex Y.

Uniqueness: If X and Y are CW complexes both of which are $K(G,1)$-complexes then X and Y are homotopy equivalent **(Hurewicz, 1936)**.

The classifying space $|BM|$

Associated with any monoid M is a combinatorial object BM called a simplicial set.

BM has n-simplices: $\sigma = (m_1, m_2, ..., m_n)$ - n-tuples of elements of M. There are face maps given by

$$d_i\sigma = \begin{cases} (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_{i-1}, m_i m_{i+1}, m_{i+2}, \ldots, m_n) & 0 < i < n \\ (m_1, \ldots, m_{n-1}) & i = n, \end{cases}$$

and degeneracy maps are given by

$$s_i\sigma = (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n) \quad (0 \leq i \leq n).$$

The geometric realisation $|BM|$ is a CW complex build from the above data which has one n-cell for every non-degenerate n-simplex of BM i.e. for every n-tuple all of whose entries are different from 1.

First attempt: F_n for monoids via $|BM|$

Fact: If G is a group then $|BG|$ is a $K(G,1)$-complex.

Since $K(G,1)$ is unique up to homotopy equivalence we have:

G is of type F_n $\quad\Leftrightarrow\quad$ $|BG|$ is homotopy equivalent to a CW-complex with finite n-skeleton.

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Definition (first attempt)

M is of type F_n $\Leftrightarrow$ $|BM|$ is homotopy equivalent to a CW-complex with finite n-skeleton.

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Definition (first attempt)

M is of type F_n $\Leftrightarrow$ $|BM|$ is homotopy equivalent to a CW-complex with finite n-skeleton.

McDuff (1979) showed that if M has a left or right zero then $|BM|$ is contractible (i.e. is homotopy equivalent to a one-point space). On the other hand, it is known that an infinite left zero semigroup with identity adjoined does not satisfy the property left-FP_1.

Conclusion

If we define F_n for monoids via $|BM|$ then $\mathrm{F}_n \not\Rightarrow$ left-FP_n.

M-equivariant classifying space $|\overrightarrow{EM}|$

Associated with any monoid M is another simplicial set $\overrightarrow{EM}$.

The n-simplies of $\overrightarrow{EM}$ are written as $m(m_1, m_2, ..., m_n) = m\tau$ where $\tau = (m_1, m_2, ..., m_n)$ is an n-simplex of BM.

The face maps in $\overrightarrow{EM}$ are given by

$$d_i(m(m_1, m_2, ..., m_n)) = \begin{cases} mm_1(m_2, ..., m_n) & i = 0 \\ m(m_1, m_2, ..., m_i m_{i+1}, ..., m_n) & 0 < i < n \\ m(m_1, m_2, ..., m_{n-1}) & i = n \end{cases}$$

and the degeneracy maps are given by

$$s_i\sigma = m(m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n) \quad (0 \leq i \leq n).$$

where $\sigma = m(m_1, ..., m_n)$.

The geometric realisation $|\overrightarrow{EM}|$ is a CW complex with one n-cell for every non-degenerate n-simplex of $\overrightarrow{EM}$.

M-equivariant classifying space $|\overrightarrow{EM}|$

M acts on $\overrightarrow{EM}$ via left multiplication.

$$n \cdot m(m_1, m_2, ..., m_n) = nm(m_1, m_2, ..., m_n).$$

$\overrightarrow{EM}$ is a free left M-set with basis BM
i.e. each element of $\overrightarrow{EM}$ can be written uniquely in the form $m\tau$ for τ in BM.

This action sends (non-degenerate) n-simplices to (non-degenerate) n-simplices, and thus induces an action of M on $|\overrightarrow{EM}|$.

Conclusion

The monoid M acts by left multiplication on the CW complex $|\overrightarrow{EM}|$. This action is free, and sends n-cells to n-cells.

Free M-CW complex

A left M-space is a topological space X with a continuous left action $M \times X \to X$ where M has the discrete topology.

Definition (free M-CW-complex)

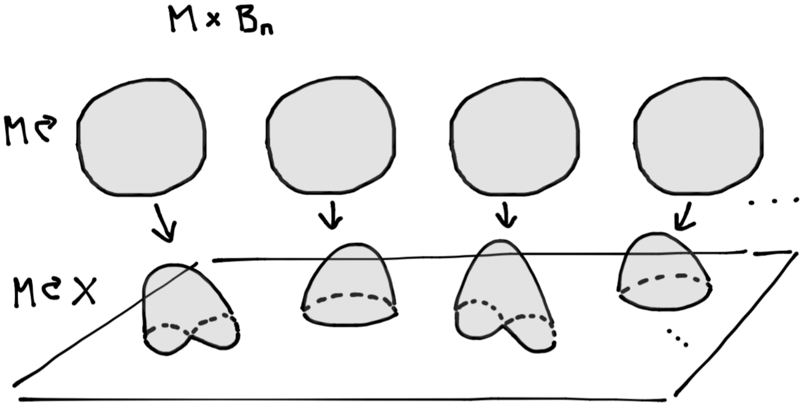
A free M-cell of dimension n is an M-space of the form $M \times B^n$ where B^n has the trivial action.

A free M-CW complex is built up by attaching M-cells $M \times B^n$ via M-equivariant maps from $M \times S^{n-1}$ to the $(n-1)$-skeleton.

X_n is obtained from X_{n-1} as a pushout of M-spaces, with M-equivariant maps, where P_n is a free left M-set.

$$\begin{array}{ccc} P_n \times S^{n-1} & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ P_n \times B^n & \longrightarrow & X_n \end{array}$$

Attaching an orbit of cells



Equivariant classifying spaces

Definition

We say that a free M-CW complex X is a left equivariant classifying space for M if it is contractible.

Existence: Every monoid has a left equivariant classifying space. Indeed, it may be shown that $|\overrightarrow{EM}|$ is an example.

Uniqueness: Let X, Y be equivariant classifying spaces for M. Then X and Y are M-homotopy equivalent.

Definition (Property F_n for monoids)

A monoid M is of type left-F_n if there is an equivariant classifying space X for M such that the set of k-cells is a finitely generated free left M-set for all $k \leq n$.

Relationship with FP_n

Proposition (RDG & Steinberg)

Let M be a monoid.

1. A group is of type left-F_n if and only if it is of type F_n in the usual sense.
2. If M is of type left-F_n, then[3] it is of type left-FP_n.
3. For finitely presented monoids left-$\mathrm{F}_n \equiv$ left-FP_n.

[3] The augmented cellular chain complex of an equivariant classifying space for M provides a free resolution of the trivial (left) $\mathbb{Z}M$-module $\mathbb{Z}$.

Finite generation and presentability

Let M be a monoid and let $A \subseteq M$. The (right) Cayley graph $\Gamma(M, A)$ has
Vertices: M Directed edges: $x \xrightarrow{a} y$ iff $y = xa$ where $x, y \in M$, $a \in A$.

Theorem (RDG & Steinberg)

Let M be a monoid. The following are equivalent.

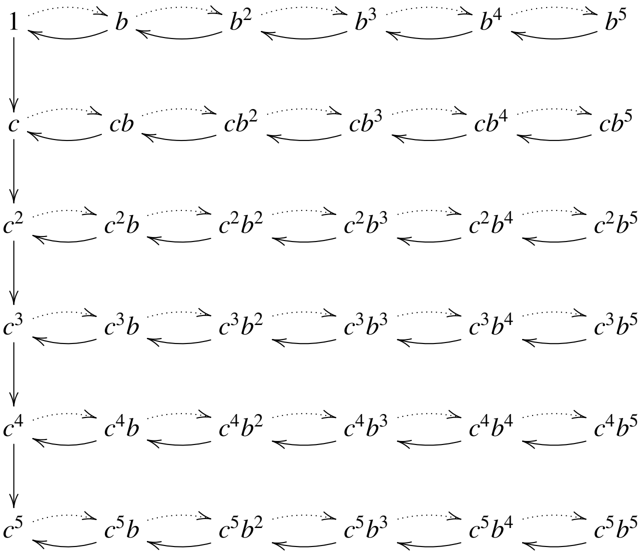
1. M is of type left-F_1.
2. M is of type left-FP_1.
3. There is a finite subset $A \subseteq M$ such that $\Gamma(M, A)$ is connected as an undirected graph.

In particular, any finitely generated monoid is of type left-F_1.

Theorem (RDG & Steinberg)

Let M be a finitely presented monoid. Then M is of type left-F_2.

Cayley graphs of semigroups and monoids

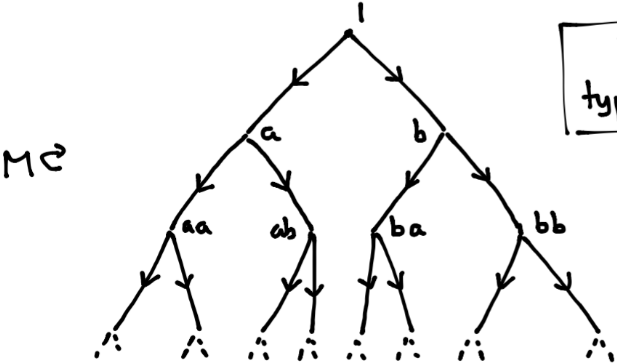


The bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$

Free monoids are left-F_∞

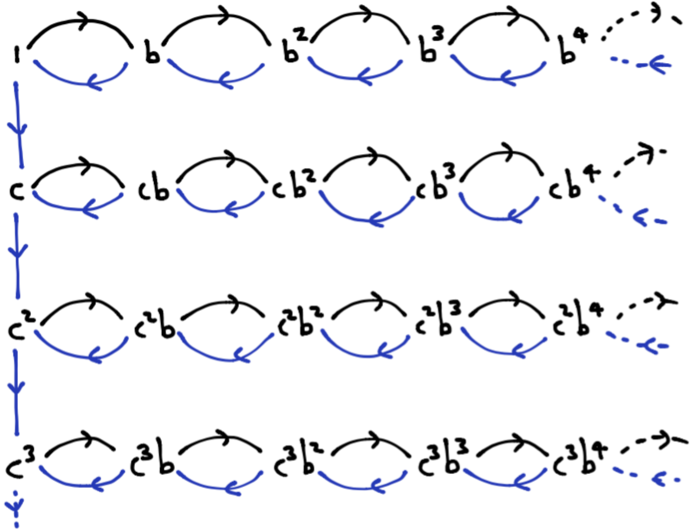
Free monoid $M = \{a, b\}^*$.

Cayley graph $\Gamma(M, \{a,b\})$

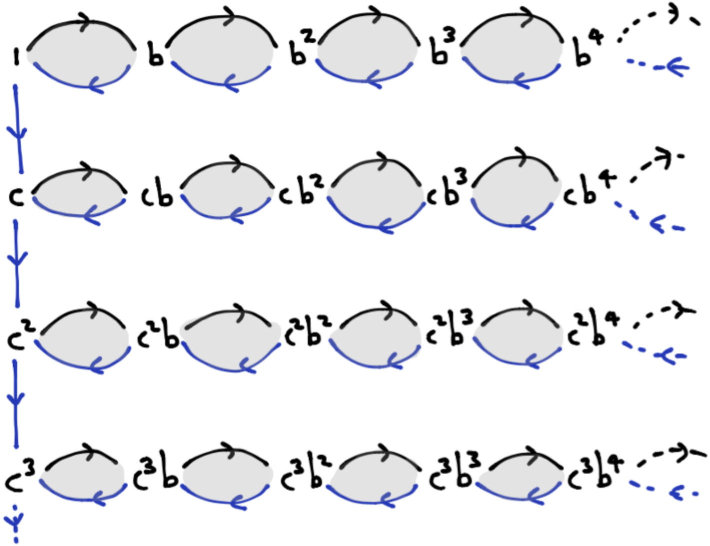


Is of type left-F_∞

Bicyclic monoid $B = \langle b, c \mid bc = 1 \rangle$



$B = \langle b, c \mid bc = 1 \rangle$ is of type left-F_∞



One-relation monoids

Question: Is every one-relation monoid $\langle A \mid u = v \rangle$ of type FP_∞?

Theorem (RDG & Steinberg)

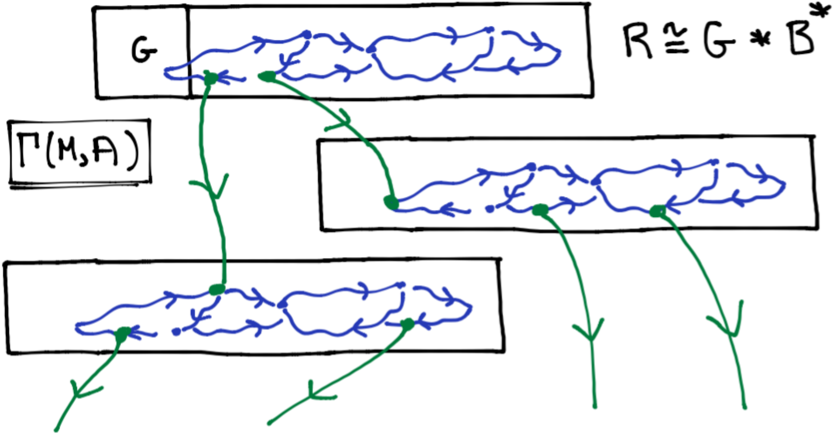
Let $M = \langle A \mid w_1 = 1, w_2 = 1, \ldots, w_k = 1 \rangle$ and let G be the group of units of M. If G is left-F_∞ then M is left-F_∞.

Corollary (RDG & Steinberg)

Every one-relator monoid $M = \langle A \mid w = 1 \rangle$ is of type left-F_∞.

Note: It is still an open whether one-relation monoids $\langle A \mid u = v \rangle$ in general are left-F_∞.

$$M = \langle A \mid w_1 = 1, w_2 = 1, \dots, w_k = 1 \rangle$$



Free products with amalgamation

A monoid amalgam is a triple $[M_1, M_2; W]$ where M_1, M_2 are monoids with a common submonoid W. The amalgamated free product is the pushout

$$\begin{array}{ccc} W & \longrightarrow & M_1 \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_1 *_W M_2 \end{array}$$

Theorem (RDG & Steinberg)

Let $[M_1, M_2; W]$ be an amalgam of monoids such that M_1, M_2 are free as right W-sets. If M_1, M_2 are of type left-F_n and W is of type left-F_{n-1}, then $M_1 *_W M_2$ is of type left-F_n.

Notes:

- The hypotheses this theorem hold if W is trivial or if M_1, M_2 are left cancellative and W is a group.
- Improves on results of **Cremanns and Otto (1998)**.

HNN-like extensions after Otto and Pride

M - monoids, A - submonoid, $\varphi\colon A \to M$ a homomorphism
Then the corresponding Otto-Pride extension is the monoid

$$L = \langle M, t \mid at = t\varphi(a), a \in A \rangle$$

Theorem (RDG & Steinberg)

Let M be a monoid, A a submonoid and $\varphi\colon A \to M$ be a homomorphism. Let L be the Otto-Pride extension. Suppose that M is free as a right A-set. If M is of type left-F_n and A is of type left-F_{n-1}, then L is of type left-F_n.

Notes:

- Is a higher dimensional topological analogue of some results of **Otto and Pride (2004)**.
- Can be used to recover some of their results on homological finiteness properties.

HNN extensions in the sense of Howie (1963)

M - monoids, A, B - submonoids isomorphic via $\varphi\colon A \to B$
$C =$ infinite cyclic group generated by t

The HNN extension of M with base monoids A, B is defined to be

$$L = \langle M, t, t^{-1} \mid tt^{-1} = 1 = t^{-1}t, at = t\varphi(a), \forall a \in A\rangle$$

Theorem (RDG & Steinberg)

Let L be an HNN extension of M with base monoids A, B. Suppose that, furthermore, M is free as both a right A-set and a right B-set. If M is of type left-F_n and A is of type left-F_{n-1}, then L is of type left-F_n.

Notes:

- This result recovers the usual topological finiteness result for HNN extensions of groups.
- It also applies if M is left cancellative and A is a group.

Brown's theory of collapsing schemes

In his 1989 paper "The geometry of rewriting systems: a proof of the Anick–Groves–Squier Theorem", Brown shows:

If a monoid M admits a presentation by a finite complete rewriting system then $|BM|$ has the homotopy type of a CW-complex with only finitely many cells in each dimension.

- To prove this he introduces the notion of a collapsing scheme.
- This idea has its roots in earlier work of **Brown and Geoghegan (1984)**.
- Collapsing schemes were rediscovered again later on as Morse matchings in Forman's Discrete Morse theory.

Brown's theory of collapsing schemes

Brown's result provides a topological proof that if G is presentable by a finite complete rewriting system then G is of type F_∞.

In his paper he goes on to say:

> *"We would like, more generally, to construct a "small" resolution of this type for any monoid M with a good set of normal forms, not just for groups. I do not know any way to formally deduce such a resolution from the existence of the homotopy equivalence for $|BM|$ above".*

Topological proof of Anick's Theorem

We have developed a theory of M-equivariant collapsing schemes which can be used to give a topological proof of

Theorem (RDG & Steinberg)

Let M be a monoid. If M admits a presentation by a finite complete rewriting system then M is of type left-F_∞.

Notes:

- We recover Anick's theorem for monoids as a corollary.
- Our results also apply in the 2-sided case and thus we also recover a theorem of Kobayashi (2005) on bi-FP_n as a corollary.

Other topics

Other topological finiteness properties

- The left geometric dimension of M to be the minimum dimension of an equivariant classifying space for M.
- geometric dimension is an upper bound on the cohomological dimension $\operatorname{cd} M$ of M.

Projective M-sets

- We actually develop the entire theory in the more general setting of projective M CW-complexes.

Two-sided theory

- We define the bilateral notion of a classifying space in order to introduce a stronger property, bi-F_n. The property bi-F_n implies bi-FP_n which is of interest from the point of view of Hochschild cohomology.